Euclid's Parallel Postulate: Its Nature, Validity, and Place In Geometrical Systems.

THESIS PRESENTED TO THE PHILOSOPHICAL FACULTY OF YALE UNIVERSITY FOR THE DEGREE OF DOCTOR OF PHILOSOPHY.

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PREFACE.

The parallel postulate is the only distinctive characteristic of Euclid. To pronounce upon its validity and general philosophical significance without endeavoring to know what Non-Euclideans have done would be an inexcusable blunder. For this reason I have given in the following pages what might otherwise seem to be an undue prominence to the historical aspect of my general problem.

In the last chapter, the positions taken are only briefly defended, because they seem to flow directly and naturally from results previously won.

I have included in the bibliography such works as are mentioned in the body of the thesis, and have not aimed at making a complete list. More complete biographies of Hyperspace and non-Euclidean Geometry are those of Halsted and Bonola, which I have mentioned in my list.

My obligations not elsewhere explicitly acknowledged are chiefly to Professor Geo. T. Ladd, at whose suggestion this study was undertaken, and under whose sympathetic direction it has attained its present form. I am also indebted to Dr. E. B. Wilson for light upon certain mathematical aspects of the problem.

New Haven, Connecticut, April, 1904.

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CHAPTER I.

THE PRE-LOBATCHEWSKIAN STRUGGLE.

I. *Historical Origin of the Parallel Postulate.*—Before the time of Euclid the science of geometry was already well advanced in Greece. Pythagoras and his immediate followers, by perfecting a theory of ratio and proportion, and by the study of areas and the introduction of irrational quantities, had brought the subject so prominently before the Greek mind that no subsequent philosopher could afford to neglect it. Accordingly, Zeno and Democritus, Anaxagoras and Hippias, Plato and Aristotle, and the great body of thinkers who were disciples of these more or less extensively devoted themselves to its study.

The duplication of the cube, the quadrature of the circle, the trisection of the angle, were all vigorously attacked and out of the struggle certain new and important conceptions arose. Certain lines of plane and double curvature were invented, important properties of conics were discovered, and the notion of infinity introduced. Methods of research and geometrical exposition were also accurately

studied, among them the method of reduction of Hippocrates, the analytical method of Plato, and the method of exhaustions of Eudoxus. Added to these the diorism of Leon, the determination by Menæchmus of the necessary conditions for the invertibility of a theorem which affords a fruitful method of enlarging the number of propositions, the introduction of formal logic and the powerful influence of the dialectics of Socrates and the Sophists, all contributed to make possible that remarkable outburst of mathematical genius which has been fittingly styled "The Golden Age of Greek Geometry."

It is obvious then in view of this remarkable development that Euclid was by no means the author of all the demonstrations contained in his Elements. It is impossible to state exactly what he did contribute. In the whole collection there is only one proof[1] (I, 47) which is directly ascribed to him. Of a few things, however, we are reasonably sure. Euclid brought to irrefutable demonstration propositions which had been previously less rigorously proved.[2] The selection and arrangement of the

[1] Gow's History of Greek Geometry, Cambridge, 1884, p. 198.

[2] Proclus, at close of the Eudemian Summary. Eudemas, a pupil of Aristotle, wrote a history of Geometry which has been lost, but Proclus, in his commentaries on Euclid, gives an abstract or summary of it, and this is the most trustworthy information we have regarding early Greek Geometry.

propositions is his.[3] He chose the theorems and demonstrations which should form a part of his system. Many available demonstrations were certainly rejected.[4] We may attribute to his deliberate choice the distinctive characteristics of the book as a whole. We owe to him that orderly method of proof which proceeds by *statement, construction, proof, conclusion,* even to the final Q. E. D. (ὅπερ ἔδει δεῖξαι) of the modern text. He is responsible too for that peculiar logical design of the book which proceeds always from a few definitions, postulates and common notions, or axioms, by sure steps which are always of precisely the same kind until every link in the argument from premises to conclusion is securely forged.

Euclid set at the beginning of his text certain definitions, postulates and common notions[5] which should serve as the foundation for his system. Was the parallel postulate one of this number?

There is but one way to answer this question, and that is to go back to the earliest editions of Euclid at present accessible and observe whether they contain it or not. In all these earlier editions there is practical agreement in regard to the defini-

[3] Proclus, Friedlein's Edition, p. 69.

[4] Compare Gow, op. cit. pp. 198 ff.

[5] Euclid did not use the term axiom. This was introduced by Proclus.

tions of Euclid. As far back as 100 B. C. Heron's[6] "Definitions" appear in the same number and in essentially the same form, though not in the same order as we have them now. Among the postulates and Common Notions,[7] however, there is considerable fluctuation. Many editions give three postulates and twelve axioms. The first nine axioms relate to all kinds of magnitudes; these remain in all editions essentially unchanged; but the last three which relate to space only and are thus distinctively geometrical fluctuate in a most interesting way. They are as follows: 10, Two straight lines cannot inclose a space; 11, All right angles are equal; 12, If a straight line meet two straight lines, so as to make the two interior angles on the same side of it taken together less than two right angles, these straight lines being continually produced, shall at length meet on that side on which are the angles which are less than two right angles.[8] In nearly all modern editions, except the most recent, these are called "axioms" and classed, as here indicated, in the same category with the other nine. In the older manuscripts, however, no such blunder is committed. It seems hardly possible to account for this

[6] Simon's Euclid Sec. 6. Die Kommentatoren des Euclid.

[7] Euclid's expression is κοιναὶ ἔννοιαι.

[8] In Clavius this is the 13th axiom. Robert Simson, on whose edition modern English texts are usually based, calls it the 12th axiom; Bolyai and many others call it the 11th axiom.

unless we assume that Euclid himself drew a distinction between them. Of these older manuscripts by far the greater number place these axioms among the postulates where they rightly belong. The Vatican Manuscript[9] gives "axiom" 10 as postulate 6. Proclus omits it altogether with the significant remark that it is really a theorem which ought to be proved, and actually attempts to prove it in book I, proposition 4. He also gives the parallel "axiom" as postulate 5, but claims that it, too, should be proved, and quotes Germinus[10] in support of this view. Thus it appears not only that this famous postulate is at least of classical origin, but also that the Greeks themselves better understood its true nature than the majority of modern writers have done.

But we have not yet answered our question; did Euclid himself actually make use of this postulate? If so, was it his own invention or did he borrow it from another? The latter question we can not definitely answer since we have no means of knowing how many of the propositions which involve this assumption were first demonstrated by Euclid him-

[9] Discovered by Napoleon in Rome in the early part of last century and brought to France. It was edited in Paris by F. Peyrard (1814-1818), who thought that he had here an edition of Euclid more ancient than Theon's (about 380 A. D.) upon which many of the early MSS. which first came to light claimed to be based.

[10] Germinus wrote about 60 B. C.

self; nothing like this postulate is mentioned however by any previous writer.

Regarding the first question we can speak with more confidence. The researches of Peyrard show that in the earliest manuscripts now accessible, this postulate does not appear among the other postulates and common notions at the beginning of the text, but is found in the demonstration of Proposition 29 where it is introduced to support the proof of the equality of the alternate angles of parallel lines. Euclid himself then not only employed this postulate but the position in which he placed it seems to indicate clearly that he appreciated the difficulties which its use involves. Surely one who had formulated a system so rigorous, who was master of a logic so keen and true that the most critical efforts of modern thought have not destroyed but, on the contrary, have only strengthened his claims to rigor, could not have passed over such a manifest begging of the question as appears upon the very face of this postulate as he himself phrased it, without having first made a desperate effort to prove it. Euclid makes no attempt as did later writers to conceal the difficulty under the cloak of a subtle phraseology. He states it frankly as a *petitio principii* of the baldest type. It must then have appeared to him not as an "axiomatic" truth, but as a theorem calling for demonstration. Euclid proves propositions mo[illegible] obvious by far. He even demonstrates that tw

sides of a triangle are greater than the third side, a proposition which the Epicureans derided as being "manifest even to asses."[11]

The position of the postulate seems to indicate that Euclid struggled on as far as possible without it and postulated it finally only because he could neither prove it nor proceed any further without it. Moreover, the astronomical system of Eudoxus and the writings of Antolycus make it also apparent that Euclid must have had some knowledge of surface spherics and was therefore familiar with triangles whose angle sum contradicts the truth of this postulate.

II. *Attempts to Dispense with the Postulate.*—The intersection of two slowly converging straight lines lies of course beyond the province of observation or construction. Hence it is obvious why the successors of Euclid, habituated by him to strict logical rigor, should have found fault with the parallel postulate and put forth their utmost endeavors to dispose of it in one way or another. In 1621 Sir Henry Saville[12] wrote: "*In pulcherrimo Geometriæ corpore duo sunt nævi, duæ labes nec quod sciam plures in quibus eluendis et emaculendis cum veterum tum recentiorum vigilavit industria.*" One of these "blemishes" was the par-

[11] Proclus, op. cit., also Cajori, History of Elementary Mathematics, p. 74.

[12] *Lectures on Euclid*, published at Oxford, 1621.

allel postulate, the other Euclid's theory of proportion. Under the title "*Parallel*" in the "*Encyclopädie der Wissenschaften und Kunste*," published at Leipzig in 1838, Sohncke says that "in Mathematics there is nothing over which so much has been spoken, written and striven, and all so far without reaching a definite result and decision." Appended to this article there is a carefully prepared list of ninety-two authors who had dealt with the problem. These quotations show the extent to which these earlier efforts were carried. Indeed it appears that almost every writer on geometry of any note from Euclid to Sohncke had given more or less attention to this difficult subject.

These earlier endeavors struck out in various directions which we shall now briefly state and consider. Some attempted to avoid the difficulty through a new definition of parallel lines; by others new assumptions which were considered less faulty were substituted for Euclid's. These in reality only concealed the difficulty; they did not remove it. A third class attempted to deduce the theory of parallels from Euclid's other postulates, by reasoning upon the nature of the straight line and the plane angle. These were by far the most desperate attempts. Finally there were those who decided that if this postulate is dependent upon the other assumptions which constitute the foundations of Euclid we shall by denying it and maintaining them,

become ultimately involved in contradiction. It was this method of procedure which resulted in the first establishment of a non-Euclidean geometry. We shall consider these attempts in the order named.

(1) *The Substitution of Different Definitions.* — Euclid's own definition was, that parallel lines are straight lines which lie in the same plane and will not meet however far produced. This definition is perhaps still best for elementary geometry. In 1525 Albrecht Dürer,[13] a German painter, proposed the familiar definition that parallel lines are straight lines which are everywhere equally distant. Clavius[14] substituted for this the assumption that a line which is everywhere equidistant from a given straight line in the same plane is itself straight. Another definition which is often preferred because of its apparent simplicity is, that parallel lines are straight lines which have the same direction. This definition possesses the peculiar advantage that those who adopt it have no further difficulty; for they find no necessity to assume the parallel postulate or anything equivalent to it. This is a great advantage, certainly; but, as a matter of fact, any one of these definitions, though apparently more advantageous than Euclid's, is in reality more complex and less satisfactory. The first two make use

[13] Cajori, p. 266.
[14] Edition of Euclid, 1574.

of the conception of distance. This of course involves measurement, which in turn embraces the whole theory of incommensurable quantities with its entire outfit of necessary presuppositions and attendant difficulties. What is more, these definitions only hold for Euclidean geometry; they are not true for pseudo-spherical space where parallel lines are still possible and where Euclid's definition is still valid. The objection to the third definition is its use of the term "direction," a word which because of its apparent simplicity, but real obscurity and vagueness, is exceedingly misleading and troublesome. For example, the straight line is often defined as one which does not change its *direction* at any point, and yet this same line is said to have opposite *directions*. Again, the angle is sometimes defined as a difference of *direction*. Motion in the circumference of a circle is said to be in a clockwise or counter-clockwise *direction*, and in this sense a point may move all round the circumference without changing its direction, and yet we speak of this same circumference as a line which *changes its direction* at every point. Killing has shown that the word direction can only be defined when the theory of parallels is already presupposed.[15]

Many other definitions have been proposed, but

[15] *Einfuehrung in die Grundlagen der Geometrie*, Paderborn, 1898.

they throw no light upon the problem, and with one exception they may be omitted. This exception is the definition proposed by Kepler[16] and Desargues,[17] which is that parallel lines are straight lines which have a common point at infinity; or "If A be a point without a given indefinite right line CD, the shortest line that can be drawn from A to it is perpendicular, and the longest line is parallel to CD."[18] This definition is important for projective geometry.

(2) *The substitution of different postulates* has been frequently made. Of these Playfair's[19] formulation that "Two straight lines which cut one another cannot both be parallel to the same straight line," is perhaps the least objectionable. Cayley[20] considered this statement to be axiomatic.

Nasir Eddin (1201-1274), a gifted Persian astronomer, in an edition of Euclid subsequently printed in Arabic and brought out in Rome in 1594, makes the following assumption: "If AB is perpendicular to CD at C, and if another straight line

[16] Kepler's *Paralipomena,* 1604.

[17] *Brouillon Proiect,* 1639.

[18] Stone's *New Mathematical Dictionary,* London, 1743.

[19] Playfair credits this axiom to Ludlum. See Halsted's article in Science, N. S. Vol. XIII., No. 325, March 22, 1902, pp. 462-465.

[20] His Presidential Address, Collected Math. Papers, Vol. XI., pp. 429-459.

EDF makes the angle EDC acute, then the perpendiculars to AB comprehended between AB and EF, and drawn on the side of CD toward E, are shorter and shorter, the further they are from CD." Or in general, two straight lines which cut a third straight line, the one at right angles, the other at some other angle, will converge on the side where the angle is acute and diverge where it is obtuse. Nothing is here said as to whether the two lines will, or will not, eventually meet; the assumption is therefore as valid for pseudo-spherical as it is for Euclidean space.

The work of Nasir Eddin was taken up by John Wallis and communicated in a Latin translation to the mathematicians at Oxford[21] in 1651; and on the evening of July 11, 1663, Wallis himself delivered a lecture at Oxford[22] in which he recommended for Euclid's postulate the assumption of the existence of similar figures of different sizes, or to quote his own statement, "To any triangle another triangle as large as you please can be drawn which is similar to the given triangle." This is easily shown to be equivalent to the Euclidean postulate. Such figures are impossible in any form of non-Euclidean space. Saccheri proved that Euclidean geometry can be rigidly developed if the exist-

[21] Wallis, *Opera* II., 669-673.

[22] Engel and Staeckel, "*Die Theorie der Parallellinien von Euclid bis auf Gauss*, Leipzig 1895, pp. 21-30.

ence of *one* such triangle, unequal but similar to another, may be granted. Carnot and LaPlace and, more recently, J. Delboeuf,[23] have proposed the adoption of Wallis's postulate.

In 1833 T. Perronet Thompson of Cambridge published a book[24] in which he brilliantly demonstrates the insufficiency of twenty-one different attempts to dispose of the Parallel postulate, and closes the volume with a demonstration of his own in which he claims to "establish the theory of parallel lines without recourse to any principle not grounded on previous demonstration."[25] This endeavor, however, belongs to the third class of attempted solutions which we must now very briefly consider.

(3) The first recorded attempt to prove the parallel postulate on the basis of Euclid's other assumptions was that of Ptolemy in his treatise on *pure geometry*.[26] This proof assumes of course the validity of postulate six,[27] which does not hold in elliptic space and also involves the untenable assertion that, in the case of parallelism, the sum of the interior angles on one side of the transversal must be the same as that upon the other side.

[23] Engel and Staeckel op. cit. p. 19.

[24] "Geometry without Axioms."

[25] Quoted from the title.

[26] Gow, p. 301. For this we are indebted to Proclus.

[27] Of the Vatican MSS.: Two straight lines can not enclose a space.

One of the most scientific attempts of this class was that of Girolamo Saccheri in his volume entitled, "*Euclidis ab omni nævo vindicatus, sive conatus geometricus quo stabiliuntur prima ipsa universæ geomatriæ principia.*" [28] This work only recently came to light. As late as 1893 Professor Klein, himself an able contributor to the knowledge of Hyperspace and non-Euclidean geometry, had not even heard of Saccheri. In 1889 E. Beltrami, at the suggestion of the Italian Jesuit, P. Manganotti, published [29] a note [30] in which he showed that Saccheri had practically wrought out a non-Euclidean geometry almost a century before Lobatchewsky and Bolyai. Apparently the only thing which prevented Saccheri from perceiving the significance of his discovery was his blinding desire to "vindicate Euclid from every fault." His statement of the problem shows clearly that he was on the right road to discovery. If the parallel axiom [31] is involved in the remaining assumptions

[28] Published at Milan, where he was president of the *Collegio di Brera,* shortly before his death in 1733. This work is now exceedingly rare, the only copy on the Western Continent, perhaps, is that of Professor Halsted, who has translated it into English. It has also been translated into German and forms a part of Engel and Staeckel's History of Parallel Theory previously referred to.

[29] *Atti della Reale Accademia dei Lincei.*

[30] *Un precursore italiane di Legendri e di Lobatchewski.*

[31] Saccheri calls it an axiom. He studied Clavius's edition, in which it appears as the 13th axiom.

of Euclid, then it will be possible to prove without its aid that in any quadrilateral ABCD having right angles at A and B and the side AC equal to the side DB, the angles C and D are also right angles and in that event the assumption that C and D are either obtuse or acute will lead to contradiction. He proves that these angles cannot be obtuse, for in that case Euclid's axiom that two straight lines cannot enclose a space is contradicted; but when he endeavors to prove that they cannot be acute he fails of his purpose for in this case he does not meet with any contradiction.

In regard to the angles C and D he distinguishes three hypotheses as follows: (1) *hypothesis anguli recti,* (2) *hypothesis anguli obtusi,* and (3) *hypothesis anguli acuti.* He then proves that if either hypothesis is true in a single case it is always true,[32] that in a right triangle the sum of the oblique angles is equal to, greater than, or less than, a right angle according as the hypothesis is *anguli recti, anguli obtusi,* or *anguli acuti.*[33] He next shows that in the first two hypotheses a perpendicular and an oblique to the same straight line will meet if sufficiently produced,[34] hence in these two cases Euclid's postulate is not contradicted.[35] He then proceeds

[32] Propositions V., VI., and VII.
[33] Propositions VIII. and IX.
[34] Propositions XI. and XII.
[35] Proposition XIII.

to prove that according as the triangle's angle sum is equal to, greater than, or less than, two right angles we have hypothesis *anguli recti, obtusi,* or *acuti,*[36] and that with the hypothesis *anguli acuti* we can draw a perpendicular and an oblique to the same straight line which will nowhere meet each other.[37]

It is unnecessary to follow him further. We now have enough to show that Saccheri understood the close connection between the parallel postulate and right angles. In his eager quest for contradictions in pursuit of the hypothesis *anguli acuti* he practically attained without knowing it, those far reaching conclusions which were disclosed a century later. But on the very verge of discovery, being blinded by an intellectual bias toward the traditional view, he rejects this hypothesis upon the unsatisfactory ground that it is incompatible with the nature of the straight line; "for," says he, "it permits the assumption of different kinds of straight lines which meet at infinity and have there a common perpendicular."

Conclusions very like the foregoing were also reached by John Henry Lambert, whom Kant calls "*der unvergleichliche Mann.*" Lambert is free from that strained reverence for Euclid which characterized Saccheri, and consequently advances be-

[36] Propositions XV. and XVI.
[37] Proposition XVII.

yond him. He starts from the assumption of a quadrilateral with three right angles and examines the consequences that follow upon the hypothesis that the fourth angle is right, obtuse, or acute. He discovers that the second and third of these assumptions are incompatible with the existence of unequal similar figures. The second assumption gives the triangle's angle sum greater than two right angles, is incompatible with the theory of parallels, but is realized in the geometry of the sphere. From this he was led to conjecture that the third hypothesis might also be realized on the surface of a sphere of imaginary radius. This is perhaps the first glimpse of the conception of "pseudospherical" surfaces afterwards developed and named by E. Beltrami. Lambert also proves that the departure of the triangle's angle sum from two right angles is a quantity which is proportional to the area of the triangle, the larger the triangle, the greater the departure. Hence in the case of elliptic and hyperbolic surfaces this angle-sum is a variable quantity which approaches two right angles as the sides of the triangle become less and less. This of course points to the fact that in the infinitesimal the spherical and pseudo-spherical triangles approach the Euclidean as a limit, and that in endeavoring to test empirically the validity of the various forms of geometry for the actual space world we must seek

among very large triangles for a measurable divergence from the Euclidean conception.

Another important suggestion which we owe to Lambert, is that in a space in which the triangle's angle sum differs from two right angles there is an absolute standard of measure, a natural unit of length.

Gauss and *Legendre* also assumed that the theory of parallels is involved in Euclid's other assumptions. Gauss did not publish anything upon the subject, and it was not known until after his death that he had interested himself in it. His correspondence, recently published,[38] shows that he was possibly in possession of a non-Euclidean system, but it does not make clear to what extent his investigations were actually carried. He announced in 1799 that Euclidean geometry would follow from the assumption that a triangle can be drawn greater than any given triangle. In 1804 he was still hoping to prove the parallel postulate. In 1830 he announces that geometry is not an *a priori* science. In 1831 he states that non-Euclidean geometry is non-contradictory, that in it the angles of the triangle diminish without limit when all the sides are increased, and that the circumference of the circle of radius r = $\pi\kappa\left(e^{\frac{r}{\kappa}} - e^{\frac{r}{\kappa}}\right)$ where κ is a constant dependent upon the nature of space. It is clear

[38] Engel and Staeckel op. cit.

from this that Gauss had in his possession the foundations of pseudospherical or hyperbolic geometry, and may have been the first to consider it as probably true [39] of the actual world. He came to regard geometry merely as a logically consistent system of constructs, with the theory of parallels as a necessary axiom; he had reached the conviction that this "axiom" could not be proved, though it is known by experience to be approximately correct. Deny this axiom and there results an independent geometry which he calls anti-Euclidean.[40] An important conception which Gauss introduced was his "Measure of Curvature," [41] which expresses the condition under which a surface has the property of free mobility of figures. This "measure" is the reciprocal of the product of the greatest and least radii of curvature and remains unchanged if the surface is bent without distension or contraction of its parts into any position. For example, we can roll up a sheet of paper into the form of a cylinder or cone without changing the dimensions of figures drawn upon it, and hence the geometry of the cylinder or of the cone is the same as that of a limited plane.

In Legendre's work there is nothing new. His results are of interest, however, because of Dehn's

[39] Compare Russell, Art. "Non-Euclidean Geometry," The New Encyclopedia Britannica, Vol. XXVIII., pp. 674 ff.

[40] *Gauss Zum Gedaechtnis*. Leipzig 1856.

[41] *Werke* Bd. IV., p. 215.

investigation which will come before us later. Legendre was able to prove, by assuming the infinity or two-sidedness of the straight line and the Archimedes Axiom of Continuity, (1) that the sum of the three angles of a triangle can never be greater than two right angles, and (2) that if in any triangle this sum is equal to two right angles, so is it in every triangle.

DISCOVERY AND DEVELOPMENT

OF

NON-EUCLIDEAN SYSTEMS.

CHAPTER II.

THE DISCOVERY AND DEVELOPMENT OF NON-EUCLIDEAN SYSTEMS.

We pass now to consider those attempts to solve the riddle which proceed upon the hypothesis that if the parallel postulate is dependent upon Euclid's other assumptions we shall by denying it and affirming them be led into contradiction. This hypothesis proved to be an exceedingly fruitful one. The absolute necessity of the parallel postulate for Euclidean geometry and the possibility of many other systems equally as rigorous and non-contradictory as Euclid itself have resulted from it. Several independent discoveries of non-Euclidean geometry have come to light. Schweikart's was the first to be published;[1] Lobatchewsky's the first to find its way into print.[2]

[1] 1812 in Charkow, communicated to Bessel, to Gerling and afterward to Gauss in 1818. See Science N. S. Vol. XII., pp. 842-846.

[2] It is interesting to note that as late as Feb. 25, 1860, a non-Euclidean geometry was worked out by Prof. G. P. Young of Canada, entirely without knowing that anything of the kind had been previously done. See Canadian Journal of Industry, Science and Art, Vol. V., pp. 341-356, 1860.

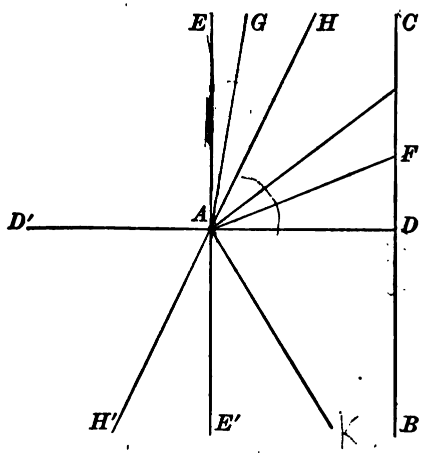
Lobatchewsky[3] defines the straight line as one which fits upon itself in all its positions so that if we turn the surface containing it about two points of the line the line does not move.

He proposes the following substitute for Euclid's postulate.[4] All lines which go out from a point in a plane may with reference to a given line in the plane, be divided into two classes — *cutting* and *non-cutting* lines. If we start in either class and move in the direction of the other we shall eventually come upon a line which is the bounding position between the two classes. This line is of course unique and is defined as being parallel to the given line. In the following figure, A is the given point in the plane and CD the given straight line. AD is perpendicular to CD and EA to AD. In the uncertainty whether the perpendicular AE

[3] Lobatchewsky first communicated his results in a lecture before the Physical and Mathematical faculty of the University of Kasan, of which he was rector, in 1826. They were printed in Russian in the Kasan Messenger in 1829 and, in much more complete and extended form, in the *Gelehrte Schriften der Universität Kasan* 1836-1838, under the title "New Elements of Geometry, with a Complete Theory of Parallels." A briefer presentation appeared in German in Berlin in 1840. Houel translated this into French at the suggestion of Baltzer in 1866, and there is an excellent English translation by Halsted, 1896. The "Elements" is by far Lobatchewsky's greatest work. This and the paper of 1829 are now both accessible in German in Prof. Engel's vol., Leipzig, 1899.

[4] Proposition 16. Paper of 1840, See Halsted's Translation.

is the only line which does not meet CD, we may assume it to be possible that there are other lines, for example AG, which do not cut DC, how far soever they may be prolonged. In passing over then from the cutting lines, as AF, to the non-cutting, as AG, we must come upon a line AH parallel to DC, a boundary line upon one side of which all lines, AG, etc., are such as do not meet



the line DC, while upon the other side every straight line, AF, etc., cuts the line DC. Now the angle HAD between the parallel HA and the perpendicular AD is called the parallel angle. If this be a right angle the prolongation AE' of the perpendicular will be parallel to the prolongation DB of CD. In that event every straight line which goes out from A, either itself or its prolongation, lies

in one of the two right angles, made by EE′ upon DD′, which are turned toward BC, so that all lines, except the parallel EE′, must intersect BC if they are sufficiently produced. In this case, which is Euclid's, there is but one line in the plane parallel to CB. But if the "parallel angle" be less than a right angle, and such an assumption is perfectly legitimate, there will then lie upon the other side of AD another line AK parallel to DB and making DAK equal to the "parallel angle."

Upon this latter assumption Lobatchewsky constructs his geometry, proving in subsequent propositions that a straight line maintains the characteristic of parallelism at every point,[5] that two lines are always mutually parallel,[6] that in a rectilinear triangle the sum of the angles cannot be greater than two right angles,[7] and that if in any triangle this sum is two right angles the same is true for all triangles.[8] He styles this system "Imaginary Geometry," because its trigonometrical formulæ are those of the spherical triangle if its sides are imaginary, or if the radius of the sphere be taken equal to $r\sqrt{-1}$.

Thus, by proving the possibility of other systems of geometry, Lobatchewsky destroys the traditional

[5] Proposition XVII.
[6] Proposition XVIII.
[7] Proposition XIX.
[8] Proposition XX.

trust in Euclid as absolute truth, and opens up a vista of new and suggestive problems; nor was he wholly unaware of the epistemological import of his discovery. He remarks, "We cognize directly in nature only motion, without which the impressions which our senses receive are impossible. Consequently all remaining ideas, for example, the geometric, are created artificially by the mind since they are taken from the properties of motion, and therefore space, in and for itself alone, does not exist for us."

John Bolyai obtained results so closely resembling those of Lobatchewsky that Russell, Klein and other distinguished writers have regarded the two as having had a common inspiration in the person of Gauss.[9] History, however, does not support this conjecture.[10] Bolyai's investigations were published as a twenty-four page appendix to

[9] Russell does this in his *Foundations of Geometry*, p. 8, and Klein, in his Goettingen lectures published in 1893, p. 175, says, "*Kein Zweifel bestehen kann, dass Lobatcheffsky so wohl wie Bolyai die Fragestellung ihrer untersuchungen der Gaussischen Anregung verdanken.*"

[10] See Halsted's Article in Science N. S. Vol. IX., pp. 813-817, June 9, 1899. Schmidt of Budapest has recently found a letter of Gauss to W. Bolyai dated Nov. 25, 1804, and accompanied by a Latin treatise on parallel lines. This communication shows that neither Gauss nor Bolyai had solved the problem. Both believed the parallel postulate to be demonstrable and were "racing to prove it."

the *"Tentamen,"* [11] a work of his father, *W. Bolyai,* in 1831, but its conception dates from 1823. He styles his new geometry the "Science Absolute of Space." The theorems necessary and sufficient for plane trigonometry in this new conception of space are the following:

(1) $\sinh\frac{a'}{k} = \sin\frac{h'}{k}.\sin A.$

(2) $\cosh\frac{h'}{k} = \cosh\frac{a'}{k}\cdot\cosh\frac{b'}{k}$, in which a' and b' are the legs of the right triangle; h', the hypothenuse; A, the angle opposite á; and k, an arbitrary constant which is presumed to be uniform throughout space. When k is infinite, finite or imaginary, these formulas give results which are true for Euclidean, Lobatchewskian, or spherical geometry respectively. K then is a form of the space constant.

Bolyai shows a profounder appreciation of the importance of the new geometry than Lobatchewsky. The latter never explicitly treats the problems of construction of the old geometry in the changed form which they must take in the new; such, for example, as, "To square the circle," "To draw through a given point a perpendicular to a given straight line," and the problem which in the new

[11] For full title see Bibliography. This work is very rare. There are only two volumes in the United States. These are in possession of Professor Halsted, to whom I am indebted for what knowledge I have of it.

geometry grows out of this: "To draw to one side of an acute angle a perpendicular parallel to the other side." All these are elegantly solved by Bolyai. He also shows that the area of the greatest possible triangle which, in this new space, has all its sides parallel and its angles zero is πi^2 where i is what we should now call the space constant.

Like Lobatchewsky he points out that Euclid is but a limiting case of his own more general system; that geometry of very small spaces is always Euclidean; that no *a priori* grounds exist for a decision; and that observation can only give an approximate answer as to which geometry is vaild for reality. Thus the new geometry casts no manner of doubt upon the geometry of perspective in so far as this deals merely with incidence and coincidence. Several propositions are equally true in all these geometries, including that of Riemann, which we are next to consider. It is mainly in the measurement of distances and angles that differences arise. In the case of Euclidean geometry the infinitely distant parts of an unbounded plane would be represented in perspective by a straight horizon or vanishing line, but according to this new geometry we cannot hold that this line would be straight; on the contrary it would be an hyperbola as in the perspective of the terrestrial horizon. If we accept Riemann's hypothesis we cannot be sure that there will be any such line at all, for we do not

know that space has any infinitely distant parts. It is possible that if we were to move off in any direction in a straight line, we might find that after traversing a sufficient distance we had arrived at our starting point.

It was not the purpose, however, either of Lobatchewsky or of Bolyai to discuss the validity of their own or of Euclidean geometry. Their motive was logical and mathematical, not epistemological or ontological. Is the result of denying the parallel postulate contradictory or non-contradictory? That was their problem. Nor did they solve it completely. The number of possible theorems in either system is practically infinite. As far as they had gone they were justified in saying there were no contradictions, but in nothing more. That latent contradictions might be revealed by further developments was perfectly possible. For this reason logical dependence or independence of any group of fundamental assumptions can never be completely tested by the method which they employed.

The purpose of *Riemann* and *Helmholtz* was a very different one. Their motive was philosophical. The attack was no longer confined to the parallel postulate. The problem was generalized. The old synthetic method of Euclid, still adhered to by Lobatchewsky and Bolyai, was now abandoned, and the properties of space were couched not in terms of intuition, but of algebra. As a result the

subsequent history of non-Euclidean geometry took on an analytical rather than a synthetic character.

Riemann and Helmholtz both sought to show that all the so-called geometrical axioms of Euclid are not *a priori,* but empirical in character. In his most remarkable dissertation[12] Riemann expresses this conviction in the following language: "The properties which distinguish space from other triply extended magnitudes are only to be deduced from experience. Thus arises the problem to discover the simplest matters of fact from which the measure relations of space may be determined. . . . These matters of fact are, like all matters of fact, not necessary but only of empirical certainty."

Riemann introduces into the problem the general conception of a manifold of which space is but a specialization arrived at through considerations of measurement. A manifold is *continuous* or *discrete,* according as there does or does not exist among its specializations a *continuous path* from one to another. When in the case of a continuous manifold we can pass in a definite way from one specialization to another, i. e., where continuous progress is possible only forward or backward, the

[12] *Die Hypothesen welche der Geometrie zu Grunde liegen.* This was his inaugural dissertation before the Philosophical faculty of the University of Goettingen in 1854. It was not published until after his death in 1867.

manifold is one-dimensional. If this group of specializations pass over into another entirely different, and again, in a definite way so that each specialization of the one passes into a definite specialization of the other, all the specializations thus formed constitute a doubly extended manifold. Similarly we may get a triply extended manifold, by supposing a doubly extended one to pass over in a definite way into another entirely different. This process may in general be repeated as often as we please, since from the analytical point of view there is nothing to limit the number of dimensions. Space, however, as known to us is a manifold of only three dimensions whose specializations are points. This we know not *a priori*, but as a matter of experience. The line is a point aggregate in which definite progress only forward or backward is possible; a surface is a one-dimensional manifold of line specializations in which progress always into new specializations is possible only forward or backward. It is therefore a two dimensional point aggregate. In a similar manner we obtain the solid as a one-dimensional surface aggregate, a two dimensional line aggregate, or a three dimensional point aggregate. But here according to experience the process stops; further progress is not possible, for we cannot in any way visualize the result.

Now to measure a continuous manifold certain postulates are necessary. Definite portions of it are

called Quanta, and comparison of these is accomplished by measuring, which requires the possibility of superposing the magnitudes compared. Therefore at least one standard magnitude must be independent of position, i. e., capable of being moved about freely without altering its value. Let us suppose that it is the length of lines which is thus independent of position and that every line is capable of measurement by means of every other. In this way position fixing becomes a matter of quantity fixing and consequently the position of a point being expressed by means of *n* variables x_1, x_2, x_3, ... x_n, the determination of the line comes in part to be a matter of giving these quantities as functions of one variable. The problem then is to establish a mathematical expression for the length of a line and to this end the quantities x must be regarded as expressible in terms of certain units. To accomplish this let it be *assumed* that the element of length ds is unchanged (to the first order) when all the points undergo the same infinitesimal motion; ds will then become a homogeneous function of the first degree of the increments of dx and remains unchanged when all the dx's change signs. The simplest case obviously is that in which ds is the square root of a quadratic function and this is the only one which Riemann especially considers.

We must now consider Riemann's conception of the "Measure of Curvature" of space which is a

somewhat obscure and misleading extension of the Gaussian conception. Since Riemann's use of this conception, especially as popularized by Helmholtz has led to much confusion, it is necessary to pause for a moment and endeavor to understand what it really means. The conception goes back logically as well as historically to our notion of the straight line as a measure of length. Since exact congruence is essential to geometrical measurement, strictly speaking, only a straight line can be measured by a straight line. If then the stretch (definite portion of a straight line) is to be the standard of all linear measurement, it is evident that we cannot measure the circle except by passing to the consideration of infinitesimal arcs, which are to be regarded as straight. Similarly our notion of curvature is referred to the circle as standard of measurement. The curvature of the circle at any point means, of course, its amount of bending or departure from the tangent line, and since this amount is constant for the same circle and is equal to the square of the reciprocal of the radius, it becomes a convenient standard for the measurement of linear curvature in general. Now since the curvature of other curves varies from point to point we again pass to the infinitesimal and measure the amount of this curvature by determining the circle which most nearly coincides with the curve at the point considered. This circle will pass through three consecu-

tive points of the curve and hence its construction is always possible theoretically,[13] for any curve, plane or tortuous. In an analogous way Gauss determined the curvature of surfaces by their amount of departure from the plane. In the case of curved surfaces we can draw through any point an unlimited number of geodesic lines whose curvature, generally speaking, will not be the same. If then we draw all the geodesics from the point at which the curvature is tested to the neighboring points on the surface they will form a singly infinite manifold of arcs among which there will be one of maximum and one of minimum curvature. Representing the radii of the osculatory circles by r and r_1 we shall have for the "measure of curvature" of the surface at the point considered the expression $\frac{1}{r r_1}$. In case of the sphere, all the geodesics going out from a point have the same curvature, and hence the surface has as its measure of curvature $\frac{1}{r^2}$. For the Euclidean plane the radii are all infinite and the curvature is therefore zero. For the cone and the cylinder, since straight lines can be drawn in each, the measure of curvature is zero and any figure not too large which can be drawn upon the plane may also be drawn without distortion of parts upon the surface of any cylinder

[13] It is interesting to note, however, that the assumption that this construction is possible is equivalent to assuming the truth of the parallel postulate.

or cone. It is therefore an obvious corollary that for any surface to possess the property of free mobility its measure of curvature must have a constant value[14] at every point.

But how is it possible to extend this conception further? In what sense is it permissible to speak of the curvature of space? Thus far it has been possible to visualize results, and in extending successively to wider and wider applications the notion of curvature we have not departed at all essentially from its original meaning. All along it has signified some degree of departure from a standard of straightness or evenness of lines or of surfaces. Does the word then still retain any vestige of this original meaning when applied by Riemann to the conception of space? In the consideration of surface curvature as given above, a third dimension of space was involved; does the curvature then of three dimensional space require a fourth dimension? These are important questions, and the failure to comprehend their meaning has led to numerous vagaries on the part both of mathematicians and philosophers.

In seeking an answer, let us note, first of all, that in the above consideration of curvature as a property of surfaces no necessary reference is made to a third dimension in the analytic development from

[14] See Minding's proof, Crelle's Journal, Vols. XIX., XX., 1839-40.

which it results. It is only when one tries to picture to himself what it may mean for sense that he seems to find need of this extra dimension. Gauss assumes that points on the surface may be determined by the two co-ordinates, u, v, and finds that small arcs of the surface will be given by the formula (1) $ds^2 = Edu^2 + 2Fdu.dv + Gdv^2$, in which E, F, and G are functions of u, v. But u, v may be lengths of lines on the surface or angles between geodesic lines and therefore we do not need to go outside the surface itself for their analytical determination. Hence the idea of a third dimension may be dropped and the measure of curvature be analytically regarded as an inherent property of the surface itself, provided of course we are dealing with a surface whose curvature is constant.[15]

Now it was this formula of Gauss which Riemann desired to make use of in determining a meaning for his "measure of curvature" as applied to space of more than two dimensions. We can see already how this conception may be given an analytical meaning. As in the case of a surface we find an unlimited number of radii of curvature at any point corresponding to the geodesics going out from the same, so in regard to space as a manifold of three

[15] Otherwise a precise co-ordinate system which is required in the development of our formula would appear to be logically impossible. Consult Russell's Foundations of Geometry, Chap. III.

or of *n* dimensions we may have an unlimited number of measures of curvature at any point corresponding to the surfaces that may be passed through it. If then we limit ourselves to a three dimensional manifold and represent its measure of curvature at any point by K we shall have $K = f\,(\chi_1\chi_2\chi_3;\ s_1s_2s_3;\ \eta, \theta)$ where $(\chi_1\chi_2\chi_3)$ give the position of the point, $(s_1s_2s_3)$ the orientation of the surface and $(r\ \theta)$ the curvature at any point on the surface. Expressing K in terms of the quantities involved in formula (1) for ds, we have,

$$(2)\ K = \frac{1}{2\sqrt{EG-F^2}}\left\{\frac{d}{du}\left[\frac{F}{E\sqrt{EG-F^2}}.\ \frac{dE}{dv} - \frac{1}{\sqrt{EG-F^2}}\ \frac{dG}{du}\right] + \frac{d}{dv}\left[\frac{2}{\sqrt{EG-F^2}},\ \frac{dF}{dv} - \frac{1}{\sqrt{EG-F^2}}\ \frac{dE}{dv} - \frac{F}{E\sqrt{EG-F^2}}\ \frac{dE}{dv}\right]\right\}$$

from which it is plain that K may have any value depending upon the meaning which we give to our linear element *ds*. An unlimited number of geometries is therefore possible, the essential foundation of any one of them being the expression which it gives for the distance between two points taken anywhere and lying in any direction from each other, beginning with the interval between them as infinitesimal. If we take $ds^2 = du^2 + dv^2$ which is equivalent to assuming the existence of a rectangle

of which ds is the diagonal and du and dv the adjacent sides, F in (1) becomes equal to zero and E and G each equal to unity. Substituting these values in (2) K becomes zero, its true value for Euclidean space.

Riemann's measure of space curvature is then simply a quantity obtained by purely analytical calculation and so far forth does not necessarily involve any relations that would have a meaning for sense-perception. What is the true meaning of *ds*? That is the important epistemological question which his treatment suggests, but does not sufficiently answer.

Riemann himself works out one form of Elliptic geometry. It is called spherical because it holds for the surface of the sphere. If the measure of curvature has in actual space a positive value howsoever small there are in reality no such things as parallel lines, for all lines meet if sufficiently produced, and space itself is limited though still unbounded.

It is clear that Riemann's whole discussion involves a profound mathematical and philosophical significance, the utmost reaches of which have not yet been fully explored.

Helmholtz and Riemann reached practically the same conclusions in regard to the nature of our space conception and the empirical origin of geometric axioms, but their methods of investiga-

tion were quite different. Helmholtz was a physicist and a physiologist, and was led to consider the problem, as he says, by attempting to represent spatially the color manifold and also by his inquiries into the origin of our ocular measure of distance in the field of vision. His method of approach was from the standpoint of natural science rather than from that of mathematics as in the case of Riemann.

While Riemann begins with an algebraic expression which represents in the most general form the distance between two infinitely near points and deduces therefrom the conditions of the free mobility of rigid figures; Helmholtz starts from free mobility as an observed fact in nature and derives from it the necessity of Riemann's algebraic expression. He also proves mathematically the latter's arc-formula, but his proof was not strictly rigid and is now superseded by that of Sophus Lie, whose attention was called to the problem by Professor Klein. Helmholtz starts with congruence, without which measurement is impossible, and then affirms that congruence is proved by experience.[16] "We measure distance between points by applying to them the rule or chain, we measure angles by bringing the divided circle or the theodolite to the vertex of the angle." "Thus all geometrical meas-

[16] "Ueber die thatsaechlichen Grundlagen der Geometrie," Wissenschaftliche Abhandlungen, Vol. II., 1866 (p. 610 ff).

urement depends on our instruments being really as we consider them, invariable in form, or at least as undergoing no other than the small changes we know of as arising from variations of temperature or from gravity acting differently at different places." In all our measuring then we can only make use of the surest means we have to determine what we are otherwise in the habit of making out by sight or touch. These statements of Helmholtz though very suggestive, do not penetrate to the heart of the matter. He is certainly right, however, in starting as he does with the facts of experience out of which our geometrical conceptions, whatever we may say as to their ultimate source and justification, have come to assume the forms in which we now have them.

It is interesting to take note of the axioms which Helmholtz comes to regard as being a necessary and sufficient basis for geometry. They are briefly as follows: (1) In a space of n dimensions, a point is uniquely determined by the measurement of n continuous variables or co-ordinates.

(2) Between the $2n$ co-ordinates of any point pair of a rigid body, there exists an equation which is the same for all congruent point-pairs. By taking a sufficient number of these point-pairs, we can get more equations than we have unknown quantities, and thus the form of the equations may be determined and all of them satisfied.

(3) Every point can pass freely and continuously from one position to another. Hence by (2) and (3) if two systems of points *m* and *n* can be brought into congruence in any position they may also be made congruent in every other position.

(4) If (n — 1) points of a body remain fixed, so that every other point can only describe a certain curve then that curve is closed.

If now we limit ourselves to three dimensions Helmholtz claims that these four axioms will be sufficient to give us the Euclidean and non-Euclidean systems as the only alternatives. This conclusion, however, is open to criticism. *Sophus Lie,* for instance, has shown [17] that the fourth axiom is unnecessary. It is included in the axiom of congruence when properly formulated. In fact, Congruence and Free Mobility are both involved in the conception of the homogeneity of space. Russell has also pointed out [18] that these four axioms flow directly from the fundamental assumption of the relativity of position.

In a second paper,[19] which from a mathematical view-point constitutes his greatest contribution to the subject, Helmholtz adds two more

[17] "Grundlagen der Geometrie," *Leipziger Berichte*, 1890.

[18] "Foundations of Geometry," pp. 128 ff, 1897.

[19] "Ueber die Thatsachen, die der Geometrie zum Grunde liegen." Wissenschaftliche Abhandlungen, Vol. II., p. 618 ff., 1868.

axioms, namely, that space has three dimensions, and that space is infinite.

The results of Riemann, Helmholtz, and Lie, had in a measure been anticipated by Wolfgang Bolyai in his famous "Tentamen," to which we have already alluded. Bolyai starts his analysis with the principle of continuity,[20] then postulates the principles of congruence and free mobility of rigid bodies[21] and finally adds to these the following postulates. (1) If any point remains at rest any region in which it is, may be moved about in innumerable ways so that any other point than the one at rest may recur to its former position. If two points are fixed, motion is still possible in a specific way. (2) Three points not co-straight prevent all motion.[22] From these assumptions he deduces both Euclid and the non-Euclidean system of his son, John Bolyai. *He also observes that the measurements of astronomy show that the parallel postulate is not sufficiently in error to interfere in practice.*[23] By casting off the assumption of the infinity of space Riemann and Helmholtz obtained as a third possibility for the universe, an elliptic geometry; but even this is suggested by John

[20] "Spatium est quantitas, est continuum." P. 442.

[21] pp. 444, Sec. 3: "Corpus idem in alio quoque loco videnti quaestio succurrit: num loca sjusdem diversa aequalia sunt? Intuitus ostendit, aequalia esse."

[22] p. 446, Sec. 5.

[23] p. 489.

Bolyai in his proof that spherics are independent of Euclid's assumption.[24]

Beltrami was the first to become clearly conscious of the fact that the theorems of Lobatchewskian geometry may be realized in ordinary Euclidean space on surfaces of constant negative curvature.[25] Minding had already shown[26] that the geometry of such surfaces so far as geodesic triangles are concerned could be deduced from that of the sphere by giving the radius an imaginary value. Beltrami, however, generalized the problem first for plane geometry in the *Saggio,* and in a subsequent paper[27] for n-dimensional manifolds of constant negative curvature. He proves that a pseudo-spherical space of any number of dimensions can be considered as a locus in Euclidean space of higher dimensions. This idea of space of one type as a locus in space of another type and of dimensions higher by one, as Whitehead says,[28] is partly due to John Bolyai. It is an exceedingly important conception because it brings into relation the geometries of Lobatchewsky, Euclid and Riemann, and

[24] See Halsted: "Report on Progress of Non-Euclidean Geometry" in Proceedings of the American Association for the Advancement of Science, Vol. 48, 1899, pp. 53-68.

[25] "Saggio di Interpretazione della Geometria non-Euclidea," *Giornali di Matematische,* Vol. VI., 1868.

[26] Crelle's Journal, Vol. XIX.

[27] "Teoria fondamentale degli spazzii di curvatura constanta" *Annali di Matematica* II., Vol II., 1868-9.

[28] Universal Algebra, Cambridge 1898, Sec. 262.

by establishing the fact that demonstrations of one geometry may by appropriate adaptation, be made to hold good for corresponding theorems in the other two it afforded, for the first time, a conclusive proof that the geometries of Lobatchewsky and Riemann are no more contradictory than is Euclid itself, and thus gave to all three geometries a co-ordinate rank.

We must now consider another very striking change, both as to method and purpose, in the historical development of Meta-geometry. Thus far we have dealt altogether with metrical conceptions. The quest has been to understand the necessary preconditions to the possibility of spatial measurement and to this end space itself has been regarded as a species of magnitude whose peculiar properties need to be defined. Starting with a doubt as to the apodeictic truth of a single Euclidean axiom, others have been called in question and the conviction has grown that they are all of an empirical and somewhat arbitrary character. One after another of those sacred postulates which for two thousand years even the best minds had considered as eternally true, was denied, and new systems of geometry sprang into existence as non-contradictory and, so far as empirical observation could go, as valid for reality as Euclid itself. Consequently that same human desire for truth, logically pure and indubitable, which for more than twenty cen-

turies had reposed with perfect confidence upon Euclid, gradually drove the geometer to seek the longed for necessity and logical purity wholly outside the realm of metrical considerations. Though avowedly mathematical and technical in its aims and purposes this new movement has attained certain results that are of far-reaching philosophical importance. Magnitude, superposition and congruence are now dispensed with and the attention is directed to the purely qualitative, as opposed to the quantitative, aspects of space.

Perhaps the greatest names connected with this movement are those of Cayley and Klein. By treating the surface of the pseudo-sphere as a plane and its geodesics (corresponding to great circles on a sphere) as straight lines Beltrami had shown that all the theorems of Lobatchewskian geometry can be developed upon this surface; but in Cayley's new Theory of Distance we *seem* to have a much simpler explanation, and one which requires no modification of the ordinary conceptions of space or of Euclidean planes and straight lines, but only an extension of the customary ideas of measurement. Cayley was throughout a staunch defender of Euclid.

In 1858 [29] he states that "in any system of geometry of two dimensions the notion of distance can be arrived at from descriptive principles alone by means of a conic called the *Absolute* and which in

[29] See Collected Mathematical Papers, Vol. V., p. 550.

ordinary geometry degenerates into a pair of points." He sets himself the task of establishing this position mathematically in his "Sixth Memoir upon Quantics" in 1859. In determining the analytic expression for the distance of two points Cayley first introduces the inverse sine or cosine of a certain function of the co-ordinates and shows that metrical properties become projective with reference to the degenerate conic called the Absolute;[30] but later he recognizes and adopts Klein's definition as an improvement upon his own.[31] This definition is expressed by the formula: distance $PQ = c \log \dfrac{AP.BQ}{AQ.BP}$ where A and B are the fixed points which determine the Absolute. This formula preserves the fundamental additive relation characteristic of distance; viz., distance PQ + distance QR = distance PR.

We can not here enter into the details of this mathematical discussion but must be content with the popular exposition which Cayley himself supplies in his Presidential Address[32] of 1883. In condensed form his conception is this.[33] Consider an ordinary indefinitely extended plane; and let us

[30] *Grundgebild,* see Klein.

[31] Collected Math. Papers, Vol. II., p. 604.

[32] Collected Math. Papers, Vol. XI., pp. 429-459.

[33] Collected Math. Papers, Vol. XI., pp. 435 ff.

modify only the notion of distance. We measure distance with a foot rule, let us say. Imagine then the length of this rule constantly changing (as it might do by an alteration of temperature) but under the condition that its actual length shall depend only on its situation in the plane and on its direction. In other words, if its length is a certain amount for a given situation and direction it will be the same whenever it returns to this position, no matter how, or from what direction it comes. Now it is plain that the distance along any given straight or curved line between any two points could be measured with this rule, and always with the same determinate result, no matter from what point in the line we begin. Of course this distance will not be what we usually mean by the term, for we do not ordinarily regard our standards as varying quantities. But for aught we know experimentally this may be what actually occurs. Suppose then that as this rule moves away from a fixed central point in the plane it becomes shorter and shorter; if this shortening takes place with sufficient rapidity, it is clear that a distance which in the ordinary sense of the word is finite, will in this new sense be infinite, for no number of repetitions of the length of the ever-shortening rule will be sufficient to cover it. There will be then surrounding the central point a certain finite area every point of whose boundary will be, according to this theory, at an infinite dis-

tance from the central point. Beyond this boundary there is an unknowable land or in mathematical language an imaginary or impossible space.

By attaching to this variable standard of Cayley's suitable laws of change, the various forms of non-Euclidean plane geometry may be had upon this Euclidean plane; and the idea may be extended so as to obtain non-Euclidean systems of solid geometry in Euclidean space.

This connection of Cayley's Theory of Distance with the various forms of Metageometry was first pointed out by Felix Klein.[34] Klein showed that if Cayley's Absolute be taken as real we get Lobatchewskian, or what he calls hyperbolic, geometry; if it be imaginary we get two forms of elliptic geometry; the double elliptic, spherical, or Riemannian already considered, in which all geodesics have two points in common; and the single elliptic which we owe to Klein[35] and in which all geodesics are closed curves having only one point in common. This is doubtless logically the simplest of all systems. If the Absolute be an imaginary point pair we get parabolic geometry, and if this point pair be what is known as circular points the result is the ordinary Euclidean system.

The natural result of all this was that both Cay-

[34] Nicht-Euclid, Book I., Chapters I. and II.

[35] Also independently discovered by Simon Newcomb. See his article in Crelle's Journal, Vol. 83.

ley and Klein came to regard the whole question of non-Euclidean systems as having no philosophical importance, since it seemed to them that it in no way concerns the nature of space, but only the definition of distance, which in their view is perfectly arbitrary, being merely a question of convenience.

Whether we accept this conclusion or not it must be admitted that the projective method, employed by them, is independent of metrical presuppositions and deals directly with that qualitative likeness of geometrical figures which is a necessary prerequisite to quantitative comparison. The distinction between Euclidean and non-Euclidean geometry is a metrical one; it essentially disappears altogether in projective geometry. Hence projective geometry deals with the conception of space from a higher point of view, which includes within its scope every variety of metrical space and whose defining adjectives must in consequence possess very great philosophical interest. Furthermore, these adjectives may also be regarded as the simplest indispensable requisites of geometrical reasoning.

In this connection the important work of Sophus Lie, which was awarded the Lobatchewsky prize Nov. 3, 1897, must briefly claim attention.[36] Felix Klein declared that this work excelled all others

[36] "Theorie der Transformations gruppen," Vol. III. Published at Leipzig, 1893.

so absolutely that no possible doubt could be entertained as to the justice of this award. Helmholtz had already originated the idea of studying the essential characteristics of space by a consideration of the movements possible in it. Klein called the attention of Lie to this problem of Helmholtz and encouraged him to undertake an investigation of it by means of his Theory of Groups. We can but meagerly indicate the outcome of this investigation. As stated by Lie his problem is: "To determine all finite continuous groups of transformations in three dimensional space in which two points have a single invariant and more than two points have no essential invariant"; meaning by invariant the distance, D, between the two points and by the statement that more than two points have no essential invariant, no invariant which is not expressible in terms of D. He finds that under the conditions of the problem the group must be six-parametered and transitive and cannot contain two infinitesimal transformations whose path curves coincide.[37] Two solutions of the problem are given. He first investigates a group in space, possessing free mobility in the infinitesimal, in the sense that if a point and any line element through it be fixed, continuous motion shall still be possible; but if, in addition, any surface element through this point and the line

[37] Transformations-Gruppen, Vol. III., p. 405 f., 1893.

element be fixed, no continuous motion shall be possible. The groups which in tri-dimensional space harmonize with these conditions Lie finds to be only those which are characteristic of the Euclidean and non-Euclidean geometries, but strange to say he also discovers that for the apparently analogous but simpler case of the plane or two dimensional space there are, besides these, certain other groups where the paths of the infinitesimal transformations are spirals. In his second demonstration, starting from transformation-equations with Helmholtz's first three postulates he proves that for a space of three dimensions the fourth postulate is entirely superfluous.[38]

From an analytical point of view Professor Hilbert in a recent article in the Mathematische Annalen,[39] which may be mentioned here, has advanced beyond these results of Lie by showing that it is possible to do away with the differentiability of functions which Lie's discussion requires. From the intuitional standpoint his article offers no improvement and is open to certain criticisms which Dr. Wilson[40] has pointed out.

[38] For a brief statement of what is essentially Lie's method in English, see Halsted's "Columbus Report," Proc. A. A. A. S., Vol. 48, 1899. Also Poincaré's Art. in Nature previously cited.

[39] Bd. 56, Heft 3, pp. 381-422, October, 1902.

[40] *Archiv der Mathematik und Physik* III. *Reihe* VI. 1. u. 2. Heft, Jan. 1903.

In his famous *Festschrift*,[41] however, Professor Hilbert has done more perhaps than any one else except certain Italians to determine the precise number, meaning and relations of the postulates essential to geometry. In this older work Hilbert followed essentially the Euclidean method with a logic so keen and pure and a result so simple that many have even expressed the opinion that it will ultimately supersede Euclid in the elementary schools. Certain defects, however, have been pointed out by Schur,[42] Moore,[43] and others, showing that Hilbert's postulates are not independent as he had supposed they were, and also illustrating how difficult it is to satisfy logic when one seeks to determine the foundations of geometry by the intuitional method. The discovery of this fact has led Hilbert in his recent article to abandon this method for the more strictly logical one. He starts from the ideas of *Manifoldnesses* and *Groups* as Lie had done, but uses the new conception of *Manifoldnesses* introduced by Georg Cantor thus dispensing with any special reference to a system of co-ordinates in a geometric space.

When the Lobatchewsky prize was awarded to Lie, the thesis of *M. L. Gérard*, of Lyons, also re-

[41] *Grundlagen der Geometrie*, Leipzig 1899.
[42] *Mathematische Annalen* Bd. 55, p. 265 ff.
[43] Transactions of the American Mathematical Society, Vol. III., pp. 142 ff.

ceived honorable mention. In this thesis Gé endeavors to establish the fundamental proposit of non-Euclidean geometry without any hypot ical constructions except the two which are assu by Euclid.[44] (1) Through any two point straight line can be drawn. (2) A circle ma described about any center with any given sec radius. But in order to establish the relations tween the elements of a triangle in a thoro going manner he adds to these, two other assu tions as follows: (1) A straight line which in sects the perimeter of a polygon in some other p than one of its vertices intersects it again, and two straight lines, or two circles, or a straight and a circle, intersect if there are points of one both sides of the other. One of the most impor considerations for the advocates of non-Euclid geometry is the requirement that all its figures s be rigorously constructed. It was to meet this quirement that Gérard's investigation was un taken and it is in this fact that its significa mainly lies.

When the Commission of the Physico-Ma matical Society of Kazan met in 1900 for the pose of awarding again the Lobatchewsky p they found before them two new treatises on n

[44] This idea was suggested and partially developed b Battaglini in his "Sulla Geometria Imaginaria di Lobatc sky," Giornale di Mat. Anno V., pp. 217-231, 1867.

Euclidean geometry, the merits of which were so nearly equal that the decision between them was finally made by the casting of lots. These were A. N. Whitehead's investigations in his "Universal Algebra" [45] and Wilhelm Killing's "Grundlagen der Geometrie." [46]

In the opinion of Sir Robert Ball, Whitehead's investigation excels anything previously done in two important particulars. In the first place he can treat n-dimensions by practically the same formulæ as those used for two or three dimensions; and secondly, the various kinds of space, parabolic, hyperbolic and elliptic, present themselves in Whitehead's methods quite naturally in the course of the work, where they appear as the only alternatives under certain definite assumptions. Perhaps the most significant portion of Killing's effort is his treatment of the "Clifford-Klein space-forms," whose importance lies in the fact that they show what a difference it makes whether we assume the validity of our fundamental axioms for space as a whole or only for a completely bounded portion of space. The first assumption yields the Euclidean and three non-Euclidean space-forms already mentioned, but the second gives a "manifoldness, at

[45] Cambridge, England, 1898.
[46] Paderborn, 1898.

present not yet dominated, of different space forms." [47]

We must now call attention to the remarkable results of Max Dehn's investigation, which was undertaken at the suggestion of Professor Hilbert and published in 1900.[48]

As early as 1898 *Friedrich Schur* had reached the conviction that elementary geometry can be built up without the use of the Archimedes axiom of continuity,[49] and proved Pascal's theorem without the use either of this axiom or of the parallel postulate. He constructs a sect Calculus in which he shows that the theory of proportion can be founded without the introduction of irrational numbers and indicates that this might also be done without the Archimedes Axiom. Hilbert accomplished this proof in 1899 and demonstrated that this axiom need no longer be regarded as necessary to elementary geometry. As we have already stated, Legendre, by assuming this axiom and also that the straight line is of infinite length, demonstrated (1) that the angle sum of any plane triangle cannot be greater than two right angles, and (2) that if in

[47] From Professor Engel of Leipzig, in a Russian pamphlet printed at Kazan, taken from Halsted's translation.

[48] Dehn was a pupil of Hilbert. He was 21 years old when this investigation was completed. It is printed in *Mat. Ann.* 53 *Band*, pp. 404-439.

[49] See Preface to his *Lehrbuch der Analytischen Geometrie*, Leipzig, 1898.

any triangle this sum is equal to two right angles, the same is true of every triangle. Hence the question arose, Do these two theorems actually hold good in Euclidean geometry? And the problem suggested for Dehn was, Can these theorems of Legendre be proved without the Archimedes Axiom? [50] The results of his investigation are very remarkable. He demonstrates the second of Legendre's theorems without any postulate of continuity, and shows that the first theorem cannot be demonstrated without the Archimedes Axiom. This is done by constructing a new geometry in which an infinite number of lines can be drawn through a point parallel to a given straight line, but in which also the triangle's angle sum is greater than two right angles. By assuming the Archimedes Axiom and also that an infinity of parallels can be drawn to a given straight line, through a given point, Lobatchewsky's geometry, in which the triangle's angle sum is less than two right angles, results; but Dehn now shows that by denying the Archimedes Axiom

[50] This axiom as stated by Dehn at the beginning of his thesis is as follows: If A_1 be any point upon a straight line between any given points, A and B, then we can construct the points A_2 A_3 A_4 . . . so that A_1 lies between A and A_2, A_2 between A_1 and A_3, A_3 between A_2 and A_4 and so forth; and moreover the sects AA_1, A_1A_2, A_2A_3, A_3A_4, . . . are equal to each other; then there is in the series of points A_2 A_3 A_4 . . . a point A_n such that B lies between A and A_n.

this angle sum is either greater than two right angles or else equal to two right angles, it cannot be less than two. He proves the former, as just stated, by his non-Legendrian geometry, and the latter case, namely, that this angle sum is equal to two right angles, by constructing another geometry in which the parallel postulate does not hold, but in which nevertheless all the theorems of Euclid are shown still to be true. He proves that the sum of the angles of the triangle is two right angles, and that various other theorems previously held to be exactly equivalent to the parallel postulate are still valid in this new geometry in which the parallel postulate is thus contradicted.

His results are well summarized by the following table:

The angle sum in the triangle is:	Through a given point we can draw to a straight:		
	No parallel.	One parallel.	An infinity of parallels.
$> 2R$	Elliptic geometry	(Impossible)	Non-Legendrian geometry
$= 2R$	(Impossible)	Euclidean geometry	Semi-Euclidean geometry
$< 2R$	(Impossible)	(Impossible)	Hyperbolic geometry

If these results of Dehn should withstand future criticism and prove to be logically unimpeachable

they will vindicate Euclid in a remarkable manner, for not one of the proposed substitutes for his parallel postulate is, after all, its exact equivalent, except when the axiom of Archimedes is already assumed.

It is impossible in this brief historical survey to do justice to certain important contributions that have very recently appeared and which from different points of starting have thrown a new light upon the foundations of mathematics in general. We refer to the contributions of Dedekind,[51] Cantor, Peano, Pieri, Padoa, Poincaré, Vailati, Russell, Frege, and others on number, continuity, series, and other topics which come to be involved in any thorough consideration of the fundamentals of descriptive, projective, and metrical geometry. Collectively considered these contributions reveal a very decided movement to carry the whole of so-called pure mathematics over to a final grounding in formal or symbolic logic. We shall refer to certain features of this movement in subsequent chapters.

It now remains to notice with a word, in closing this chapter, the efforts that have been made to deal with the philosophical problems created by metageometry. These efforts have proceeded sometimes

[51] Some of these writings are in reality not so recent as some of those which we have already considered, but they all belong to the one general movement to which we wish to call attention.

from the mathematicians themselves, as in the c
of Riemann and Helmholtz, and sometimes fi
students of philosophy. With one notable exc
tion they have all suffered more or less from
writer's inability to take the point of view of
philosopher on the one hand or of the mathematic
on the other — an incapacity due to lack of spe
training, to see the problem clearly and steadily
both its mathematical and its philosophical relati
The exception referred to is that found in the c
tributions of B. A. W. Russell, especially in
Principles of Mathematics,[52] the first volume
which only recently appeared. Mr. Russell bri
to his study of the problem the training both c
mathematician and a philosopher; the result is a c
tribution of remarkable and permanent value.

In view of the historical development thus so
what imperfectly presented, we shall endeavor in
chapters which follow, (1) to orient the prob
and point out its complex relations; (2) to tı
the parallel postulate and its closely allied con
tions to their psychological sources; (3) to de
mine the nature and validity of this postulate and
place in geometrical systems; and finally (4) to
dicate the conclusions which seem to follow fı
this discussion as to the nature of space.

[52] Published at Cambridge, England, 1903. This is bey
question the best work on the Philosophy of Mathematics
published.

GENERAL ORIENTATION

OF THE

PROBLEM.

CHAPTER III.

A GENERAL ORIENTATION OF THE PROBLEM.

The foregoing historical sketch brings prominently to view certain matters of great philosophical interest. First of all it has certainly become clear that in so far, at least, as any system of geometry has professed scientific value, or has claimed to be in any sense valid for reality, the whole history of meta-geometry has been, as a matter of fact, one long and very fruitful search for the philosophic foundations of mathematics in general. The same spirit which through the centuries endeavored so earnestly to justify Euclid as an orderly system of necessary and indubitable knowledge by removing the objectionable theory of parallel lines, has finally subjected the fundamentals of arithmetic as well as those of geometry, to the most searching critical testing. The sufficiency, independence, and mutual compatibility of the various adjectives which presumably define our notions of number and space, have become problems of absorbing interest and promise. As is usual in every marked intellectual advance, every existing difficulty removed has

opened up new fields of research, new tendencies of thought and methods of investigation, and consequently new and more difficult problems calling for solution.

The light thus thrown both directly and indirectly upon the space problem has led to a very great refinement of the space conception which has resulted more and more in restricting the *a priori* realm and in handing over to the empirical, as possibly contingent and depending ultimately upon the peculiar nature of experience, certain matters which were previously thought to be apodeictically true. Prior to Lobatchewsky "geometry upon the plane at infinity" was considered as being just as well known as the geometry of any portion of the table upon which I am writing, but today the geometer "knows nothing about the nature of actually existing space at an infinite distance; he knows nothing about the properties of this present space in a past or a future eternity."[1] He does know, however, that, within the limits of the utmost refinements of instrumentation and observation thus far attained, the assumptions of Euclid are true for small portions of space and perhaps, when all due allowance for probable error is made, even for that immense region which

[1] W. K. Clifford: *Lectures and Essays*, Vol. I., p. 359. London, 1901. We do not subscribe to the naive space-realism latent in these words of Clifford; the passage is quoted because it indicates very clearly the changed point of view regarding the nature of geometry.

is swept by telescopic vision. Hence the important question as to what are the necessary and sufficient marks of the category of space, once regarded as settled, takes on decidedly a new interest for speculative thought.

Glancing back for a moment over the history of this movement, one can easily trace from a psychological view-point the predominant intellectual and practical interests out of which it has grown. It is often contended that geometry is concerned with ideal objects. At present, this is certainly true; but as a matter of history it has not always been true. Geometry is in reality a complex product of two factors; the one empirical or, if you please, intuitional, and the other logical. Both appear to be necessary to any geometry which would validate its claims to be a *bona fide* body of systematic knowledge. Our historical sketch shows that these factors have been variable quantities, the intuitional element having been steadily reduced until at present so far as space is concerned it is entirely rejected. Geometry as a part of "pure" mathematics is coming to be regarded as merely a branch of symbolic Logic, which no longer claims to throw direct light upon the nature of space.

Nevertheless, in contemplating the "pure" abstract science which thus claims to be free from all intuitional bias, we should not forget its humbler

origin. Even among the Egyptians [2] and the early Greeks,[3] where authentic history first finds the subject already somewhat advanced, it is almost wholly an empirical matter. In the hands of the later Greeks, however, the treatment of geometry underwent essential modifications. This naive conception of things disappears. Geometry passes from a purely technical to a scientific state and becomes the subject of professional and scholarly contemplation. For the first time a conscious effort is made to separate the directly cognizable from what is logically deducible and to throw into distinct relief the thread of deduction. For purposes of instruction the principles which are simplest, most easily gained and apparently freest from doubt and contradiction are placed at the beginning, and the remainder based upon them. The motive now arises to reduce the number of these principles as far as possible. In this respect the superiority of Euclid was recognized, as we have seen, for more than twenty centuries. His selection of fundamental conceptions seemed to withstand every opposition and all efforts at a further reduction.

Having at length found, however, that the denial of Euclid's parallel postulate led to a different sys-

[2] Compare Eisenlohr: *Ein Mathematischen Handbuch der alten Aegypter: Papyrus Rhind,* Leipzig, 1877.

[3] James Gow: A Short History of Greek Mathematics. Cambridge, 1884.

tem which was self-consistent and possibly true of the actual world, a new motive arose. "Mathematicians [4] became interested in developing the consequences flowing from other sets of axioms more or less resembling Euclid's. Hence arose a large number of geometries inconsistent as a rule with each other but each internally consistent." Even the resemblance to Euclid at first required in any set of axioms which it was desired to investigate was gradually disregarded. Possible systems were investigated on their own account, and thus, as we have said, the intuitional aspect of geometry became altogether a matter of indifference. Geometrical propositions are no longer assertorical in character. They do not claim to state what actually is, but merely assert that certain consequences flow from given premises. Whereas Euclid asserted not only that certain geometrical inferences were logically sound but also that both the premises and the conclusion were actually true, the new geometry pronounces upon the inference merely, and leaves premises and conclusion both as matters of doubt. The implications alone belong to geometry; with axioms and propositions it is not concerned. The geometer deals with certain entities to be sure, but these are carefully defined and guaranteed to exist only in the sense that they are logically compatible.

[4] Hon. B. A. W. Russell's "Principles of Mathematics," Vol. I., p. 373. Cambridge, 1903.

They are not even necessarily points, or lines, or any of those objects usually regarded as the legitimate subject-matter of geometry, but may be any mental constructs which harmonize with certain conditions arbitrarily chosen. The question as to whether any set of axioms and propositions hold of actual space or not, is then a problem of applied mathematics, to be decided, so far as decision is possible, by experiment and observation. Pure mathematics contents itself with merely asserting that *if* any space has such and such properties it will also have such and such other properties.

Riemann's generalization through the introduction of analytical conceptions so extensively employed by subsequent writers and the important works of Dedekind,[5] Cantor,[6] and others, on the nature of continuity, have given rise to new interests in line with the general demand for logical rigor and have shown the necessity of subjecting the prerequisites of analytical geometry to a careful investigation.

In the employment of the analytical method space was regarded as a manifold of points referred to a system of co-ordinates and each capable of being definitely determined by means of numbers. Lines were defined by establishing a one to one correspondence between the ensemble of numbers and a certain

[5] "Was sind und was sollen die Zahlen." Brunswick, 1893.
[6] "Ein Beitrag zur Mannigfaltigkeitslehre," Borchardt's Journal, Band 84, pp. 242-258, Dec., 1877.

series of points; surfaces, by setting up a similar correspondence between the ensemble of numbers and a series of lines; solids, by establishing the same correspondence between the ensemble of numbers and a certain series of surfaces. Thus the geometrical continuum came to be regarded as generated by the number of continuum, and the question naturally arose as to how this course of procedure may be justified. Dare we include all numbers in the ensemble spoken of, imaginary and real, and rational and irrational? In the development of geometry this has actually been done. It is indeed "impossible to exaggerate the importance even of imaginary numbers, for without them the fabric of modern geometry could not stand for a moment.[7] Hence in investigating the relations of Euclid to the modern system, especially if we regard the latter as the more ultimate and "pure," the question of right becomes an interesting one. Is there not really, though perhaps not so patently, as much room for doubt here as in the case of the parallel postulate? Given a co-ordinate system, we may readily admit that *if* any set of quantities actually determine a point, they determine it uniquely, but *how do we know that they determine it at all?* That is the interesting question. To say that to each of a certain series of objects there corresponds a number is

[7] Professor Edwin S. Crawley, Popular Sci. Mon., Jan., 1901.

surely quite different from saying that to every possible number there corresponds an object in the given series. Cayley's Absolute requires for Euclidean geometry circular points at infinity. Dare we assume that there is anything in reality corresponding to these mythical entities? Not pausing to reply but only to indicate the general course of this interesting development, we need only say that if this question be answered affirmatively there arises another no less interesting and difficult, which has contributed its share to the quest for mathematical rigor resulting in the general thought movement characterized by Professors Pierpont[8] and Klein as the "Arithmetization of Mathematics." Rational numbers apparently give no trouble. Their arithmetic is easy, but when it comes to interpolating among these that infinity of irrational numbers, such as radicals, logarithms and others, met with in the development of mathematics, it has been customary to proceed upon the tacit assumption that the arithmetic of these is the same as that of rational numbers and that whatever operations may be performed upon the one class may also be performed upon the other. Thus the Kantian "*Quid Juris?*" again confronts us. Certain eminent mathematicians have replied that no rigor is possible except upon a basis of rational numbers.

[8] Bull. Amer. Math. Soc., 2d series, Vol. V., No. 8, pp. 379-385. May, 1899.

To meet the demands of such reasoning the old Aristotelian logic was, of course, not altogether adequate. It, too, like Euclid, stood in need of critical renovation and extension. To get rid of the want of accuracy which creeps in unnoticed through the association of ideas and is therefore not allowed for when ordinary language is employed, symbols for different logical processes were introduced. It was found that all deduction is not syllogistic as the scholastics had thought. Asyllogistic inferences must also be recognized. A new logic was created, for which Boole, C. S. Peirce, and Peano have been largely responsible. To this final court of appeal all mathematical difficulties are henceforth to be brought. Russell's latest work is an elaborate and thorough-going effort to establish the thesis that all pure mathematics, geometry included, is merely a branch of this symbolic logic.

All this then, as inspired by one supreme motive, an age-long struggle for what men have seen fit to call ***absolute rigor***. If we turn to the cautious mathematician and ask what he means by this word, how he shall know when he has attained it, and what is his standard, we do not find him at all ready to reply. Indeed it is wise to be silent, for much that was once thought to be rigorous is now no longer so regarded; a large part of the reasoning of the last century would be rejected today. An English or American treatise on Calculus twenty

years old is now almost as obsolete as a work on chemistry of the same date would be. Geometrical rigor is a variable quantity approaching a limit which can scarcely be reached except as mathematics becomes wholly divorced from actual sense experience. But geometry refuses to be thus a mere matter of logic. So regarded its territory can only be arbitrarily defined. It becomes a mere study of multiple series, so Russell has actually defined it, and as such it includes complex numbers as a legitimate part of its subject-matter. But why restrict it to multiple series, why not also include series of only one dimension, and thus do away with the name geometry altogether. Indeed it is only when we introduce some notion of applied geometry, some conception so defined as to resemble more or less approximately what we know to be true of our actual space, that our employment of this term seems to be anything more than an unjustifiable misuse of words. Geometry as logic may indeed care very little as to the particularized existence, either actual or possible, of the entities with which it deals, but geometry, as *geometry*, in any justifiable meaning of this word, is certainly something more than this. And even as pure logic it is in a sense still subject to the space category; its entities are conceived as delimited, externalized, and otherwise spatially related. They are supposed to be capable of certain spatial transformations. How, then, do

we know that certain positions and transformations of these entities are allowable, except as we fall back upon those more ultimate axioms which though they have a wider application are still in an important sense geometrical. They are logical necessities which cannot even be thought, or regarded as having any meaning whatever, without reference to some sort of co-existent realities actual or abstractly possible which are conceived of as entering into relations with each other which are only partially defined by these "axioms" and which inevitably imply the space category.

But upon the consideration of the logical and epistemological questions thus brought to view it is not our purpose at present to enter. The aim here is simply to point out the motive dominating this movement and to show that the general problem which lies before us is one of extreme complexity, which has been by no means generally understood by those who have essayed its solution.

Certainly the profound questions growing out of the parallel postulate, its validity for reality, the general nature of space which this involves and the fundamental relations of Euclid to other systems of geometry can never be settled by pure mathematics alone, remarkable as the contributions from this source have certainly been. The non-Euclidean movement, *as such*, has probably produced already almost all the modifications it is likely to produce in

the foundations of geometry.[9] In the final answer, if indeed a final answer is possible, the psychologist, the philosopher, the physicist, and the physiologist must each have something to say. The claims of individual investigators to have solved the problem completely, and such claims even yet occasionally appear, can only be looked upon with a measure of suspicion.

On the philosophical side of the problem the old question as to the *a priori* nature of the postulates of Euclid affirmed by Kant and denied by Mill, is still in debate. The psychological analysis necessary to a just and fruitful treatment of this question has at times been entirely wanting; at others, its significance has been utterly ignored. The term *a priori* itself has been used in widely different senses by different writers and sometimes even by the same writer. The word "curvature," as applied to space, unfortunately introduced by Riemann, Helmholtz, and Beltrami, has also been a source of perennial confusion for both mathematicians and philosophers alike. The essential meaning of the straight line as used in the various forms of geometry, and of the subordinate conceptions of distance, direction, and motion by which this line is usually defined, has not been clearly determined. And finally the abstract spaces of geometry demand further study as regards

[9] Consult Russell's "Principles of Mathematics," p. 381.

their relations to each other and to the space of actual experience. They require to be more thoroughly tested by that conception which must arise when all *bona fide* human experiences of a spatial order are taken into the account.

In the next chapter we enter upon a psychological study of the parallel postulate and certain closely allied conceptions without which this postulate would have no meaning at all. Our purpose will be twofold. First, starting with experience, we shall endeavor to trace the genesis and development of these conceptions as they appeal to the ordinary consciousness of man; and secondly, starting with these conceptions in the highly abstract forms demanded by modern geometry, we shall try to see how far these forms may be made to have meaning for actual experience.

THE PSYCHOLOGY OF THE PARALLEL POSTULATE

AND ITS

KINDRED CONCEPTIONS.

CHAPTER IV.

PSYCHOLOGICAL SOURCES OF THE PARALLEL POSTULATE AND ITS CLOSELY ALLIED CONCEPTIONS.

There can be little doubt that geometry sprang originally from man's interest in the spatial relations of physical bodies. It bears in every part unmistakable evidence of this empirical origin, and the course of its development can be rendered fully intelligible only on consideration of these traces. Various forms of sensory experience contributed the data. By virtue, it may be, of the peculiar structure of the body with its pairs of sense-organs symmetrically located, whose conscious deliverances possess in each case a remarkable similarity as contrasted with those of other sense-organs, the power of orienting these organs themselves and finally of the whole body with reference to presented stimuli, has at last been acquired.[1] This power of orienta-

[1] Consult Loeb's *Physiology of the Brain* (N. Y., 1900). Also Royce's *Outlines of Psychology*, pp. 139-147. It is perhaps due to this cause that the lower animals know how to strike in and to hold more or less steadily a straight direction in movement. Consult Von Cyon's articles in Pflueger's Archiv fuer Physiologie.

tion, taken with sensations of movement,[2] sight, and touch,[3] and of the so-called statical sense of the semicircular canals, furnishes the empirical basis for the perception of space as a continuous whole.

The unitary perception of space thus arising is, of course, complex in its nature and is determined in each case by the character of the sensory factors which it actually involves. So-called psychological spaces corresponding to different senses are not wholly identical. The space-perception of a man born blind [4] is unitary in character, but quite different from that of a man whose vision remains unimpaired. These sensory differences have been unconsciously carried over into the foundations of geometry, so that the different forms of this science can be classified as motor, visual, etc., according as special emphasis has been laid in their construction, now on one, now on another, of these sensory factors. Projective geometry is entirely visual, while Euclid is largely motor. These psychological distinctions are important when questions of validity are raised regarding the foundations of competitive systems. Had man's spatial experience been con-

[2] Poincaré holds that without sensations of movement and the *actual ability to move* geometry could never arise. *The Monist.* See also Professor Ladd's *A Theory of Reality,* pp. 229-230.

[3] Ladd's *Psychology Descriptive and Explanatory,* pp. 323 f.

[4] Consult Dr. Alexander Cameron's Thesis on "Tactual Perception." Yale, 1900.

fined, for example, to vision[5] alone, the struggle between Euclid and Lobatchewsky could never have been, since for vision alone there are no such things as parallel lines. Through a point in a plane there is neither one nor a pencil of lines which do not cut a given line in the plane; every such line is seen to converge toward the given line in one or in both directions.

When, therefore, we come to inquire what are the sensory factors that enter into the subordinate conceptions which define the spaces of the different geometries, where in each the emphasis is actually laid, in the choice of conceptions and in the character of the demonstrations which follow upon them, and finally what is the true balance to be maintained among these sensory factors in determining the nature of space as it appears to be actualized in the world of reality, it is plain that the psychological difficulties involved strike deeper than mere questions of accommodation and convergence in visual perception. Nevertheless, investigations of this sort have thrown, and no doubt will continue to throw, light upon the general problem. We organize our experiences to harmonize with our own and with the movements of physical bodies; consequently those changes in the environment of an object which necessitate changes in the character of the move-

[5] This is perhaps an impossible hypothesis, but it serves our purpose in this connection.

ments by which it is perceived occasion differences in the perception of its spatial properties. In this way optical illusions arise. With these so-called false perceptions, eye-movements [6] are now known to be closely related if not indeed a determining cause. The fact that long-continued practice dispels these illusions, the perceived object remaining constant while the perception itself gradually but unconsciously changes, indicates that the peculiar marks of any concept of space founded simply upon visual perceptions can hardly be called *a priori* in Kant's sense of the word. Photographs taken at intervals during the presence of these optical illusions and after they have finally disappeared show quite clearly that changes in eye-movements corresponding to those in the perception itself successively occur. Increasing accuracy of movement and correctness of perception develop together. Whether the movement or the perception itself is to be regarded as the determining cause of this improvement is an interesting but difficult question. It seems very probable, however, that cerebral organization and accurate motor adjustment must first be secured before correct perception becomes at all possible. If the mere recognition of error were all that is needed, no long-continued process of perceptual edu-

[6] Based upon some very interesting but as yet unpublished experiments in photographing eye-movements in the Yale Laboratory by Assistant Professor Judd and Dr. McAllister.

cation would seem to be required. Correct perception would then take place as quickly and completely as when, through a false perception, we have mistaken some stranger for a friend.

Turning now from these general considerations to study those special facts from which geometry as a science has actually developed, we find that the first geometrical knowledge which can be strictly so-called was acquired accidentally and without design through practical experiences in varied employments. This peculiar empirical origin is shown in what history we have of the beginnings of geometry among the ancient Egyptians [7] when as yet the scientific spirit in its search for the logical interconnections of the experiences in question had not arisen. It appears also even more clearly in the history of primitive civilizations at large, as may be seen in the rise of metrical conceptions and in those facts of experience out of which the conception of the straight line and Euclid's theory of parallels have developed. We shall now study these developments in the order here mentioned.

Primitive man was already well advanced in certain geometrical ideas before measurement, strictly so-called, began. He had acquired a considerable practical knowledge of physical bodies and their grosser spatial relations without taking advantage

[7] Consult Gow's A Short History of Greek Mathematics, Cambridge, 1884.

of this artificial assistance. Through the comparison of various kinds of sensory experiences he had come to attribute to bodies a certain spatial constancy; he had learned to locate them, to estimate their form, size, and distance, with considerable accuracy and to govern his actions accordingly. But it was in the objects themselves, in their capacity to satisfy his needs and not in their spatial relations that his interests primarily centered. It was only when he had so far triumphed over his foes in the struggle for self-preservation as to be able to reflect that the place, form, and size of objects have almost everything to do with determining the character of those activities by which his wants are best satisfied and his enemies overcome or avoided, that the desire for a more accurate determination of these quantities by means of measurement arose. His first estimates were, of course, obtained by the comparison of memory images with present perceptions. This mode of estimation, however, depends upon certain physiological and psychological conditions which are difficult to control and is therefore unsatisfactory when exact measurement is required. This is especially true when the interval of time between the experiences compared is large and the memory image has, as a consequence, considerably faded. Hence it becomes necessary to provide characteristics which depend as little as possible upon these conditions, and this can only be done by removing the time interval

between the remembered and the perceived experiences altogether, or by rendering it as brief as possible,— the ideal being the substitution of direct perception in the place of memory. This can be done only by securing the exact congruence of the bodies compared, and hence it is always theoretically impossible, because of the inevitable limitations of sense-perception, and the want of absolute rigidity in all natural objects. Nevertheless this is the principle of measurement, and it remains the same for all spatial measurements whether performed by the lowest savage or the most exact geometer, as a direct perception of physical congruence or as a purely abstract visualization.

For all physical measurement, then, a convenient portable standard of some sort is needed, and one whose want of appreciable variation during transportation may be directly perceived. Naturally the first objects of this sort to appeal to the primitive man would be various parts of his own body. The names of the oldest measures, and various other facts, indicate clearly that these were the standards actually employed. Among these names, for example, are the *hand, nail, ell, span, cubit, foot, fathom, pace,* and *mile.* These names have, of course, lost much of their original meaning, and have now come to be associated with standard measures which they happen to represent with only tolerable accuracy.

It was a significant forward step in civilization

when nations like the Egyptians and Babylonians abandoned these physiological for more exact physical standards. But the evolution of this form of measurement has also taken place in accordance with known psychological laws. In harmony with the general principle that it is in the material rather than in the more distinctly spatial properties of objects that human interests first become centered the first measurements of this sort were doubtless measurements of volume,[8] not measurements by means of definitely chosen standards of volume, however, but merely inaccurate estimates of the capacity of vessels and storerooms by determining the quantity or number of similarly shaped bodies which they would contain.[9] Similarly the first estimates of area were probably made by the number of fruit-bearing trees which a field would grow, the amount

[8] It is interesting to note that those who have given most attention to the best methods of introducing geometry to beginners have generally agreed that it is best to begin with solids and considerations of volume, rather than with points, lines, and angles, and there is a decided movement now on in elementary education in America to follow with the individual a method very similar to that by which as we here find the race has been educated. Note some recent American publications such as Campbell's *Observational Geometry*, New York, 1899; Spear's *Advanced Arithmetic*, Boston, 1899; Hanus's *Geometry in the Grammar School*, Boston, 1898. The Harvard Catalogue for 1901-1902, p. 307; Row's *Geometric Exercises in Paper-Folding*, Chicago, Open Court, 1901—Tr.

[9] E. Mach: *The Development of Geometry*, The Monist, Vol. XII., p. 486. Also Eisenlohr: *Papyrus Rhind*, etc., previously cited.

of labor necessary to cultivate it, or the number of animals that could be grazed upon it. The measurement of one surface by another may have been suggested by estimating in this way the relative value of fields which lay near one another.

Herodotus [10] states that when Xerxes wished to count the army which he led against the Greeks he adopted the device of drawing up 10,000 men closely packed together within an enclosure which was then made to serve as a standard, and each succeeding division that filled it was counted as another 10,000. This is an example of another type of operations which naturally led to the measurement of one surface by another. By abstracting at first instinctively and then consciously from the height of the practically identical bodies thus covering a surface the notion of a surface unit would finally be reached. The fact that among the Egyptians,[11] the early Greeks, and even as late as the Roman [12] surveyors' rules for the measurements of surfaces of irregular figure were often grossly inaccurate, seems in general to favor this view.[13]

But what needs to be emphasized most perhaps in

[10] Herodotus VII., *22*, 56, 103, *223*.

[11] Eisenlohr: *op. cit.*

[12] M. Cantor: *Die romischen Agrimensoren*, Leipzig, 1875.

[13] Thucydides VI., 1 states that surfaces having equal perimeters have equal areas. And Ahmes in *Papyrus Rhind* gets the area of the triangle by multiplying together two of its sides.

connection with measurement is the fact that all acts of measurement exhibit the free-mobility of approximately rigid bodies, and that these facts when conceptualized bring to clear consciousness the corresponding postulate which lies at the foundation of every system of metrical geometry. And furthermore the recognized possibility of constructing similar solids of different sizes leads by a similar pathway to the postulate of homogeneity.

The modern method of defining points, lines, and surfaces as boundary conceptions, though logically necessary and presenting little difficulty to a mind skilled in abstract thinking, nevertheless conceals rather than exhibits the process whereby these conceptions have developed. The straight line still bears in its name a suggestion of its origin. *Straight* is the participle of the old verb *to stretch* and *line* is from *linen,* which signifies a *thread,* hence the *straight line* literally means a *stretched cord* or *thread.* By decreasing the thickness of such an object until it becomes vanishingly small the conception of the line as a geometrical magnitude of only one dimension is reached by an easy abstraction.

By making fast one end of a string and drawing the other through a hole or a ring and observing how more and more of it passes out until the whole becomes *stretched* or *straight* we have an example of that type of experiences from which the notion of

the straight line as the shortest distance between two points came to be clearly discerned.

Another class of simple experiences supplies that peculiar property of the straight line by which it may be defined as a unique figure determined by any two of its points. If we slide a crooked material line between any two fixed points it will be observed that the portion between the two points will continually alter as regards the form and position of its parts. The more uniform the object, the less this variation becomes until the limit of perceptual uniformity is reached, when the object will be seen to slide within itself. We now have a property which belongs obviously as much to the circle or the spiral as it does to the straight line, for these figures also when thus operated upon are seen to possess it. But if we rotate these objects about the two points in question a difference is at once detected, and we come upon a peculiar property of the straight line. *It rotates within itself.*

Now when we consider these two properties of the straight line, *that it rotates within itself and is also the shortest distance between two points,* it becomes perfectly obvious why this figure has been made fundamental in all systems of geometry. *It is the only one-dimensional magnitude that is physiologically simple and perceptually constant when viewed from any point not in it.* The same is true of the plane, and explains its unique position among

two dimensional magnitudes. It is the perceptual simplicity and constancy of these two figures that has forced them upon us as the only invariable, and hence the only satisfactory, metrical standards. Among tridimensional objects the cube and the sphere hold a similar relation, and for a similar reason. Of these two figures the sphere is of course the simpler from a perceptual point of view. It alone of all solids appears constant, in form, from whatever external point we view it; nevertheless the law of simplicity and economy holds in the selection of the cube as the fundamental standard of volume because of its obvious relations to the one and two dimensional standards which we have just considered.

Any theory of parallel lines must obviously involve the idea of surfaces, hence before passing on to a direct study of Euclid's conception of parallels it is necessary to take some note of the origin of this idea. The conception of surfaces as geometrical magnitudes absolutely without thickness was certainly not reached at a bound. Nor is it a notion which is rendered necessary by the essential nature of a perceiving mind. It is essentially a metrical conception, and is therefore conditioned by those characteristic of things and our experiences with them which render measurements possible. Like the conception of spatial equality, it could have no meaning in a homogeneous fluid world, and in such

a world would doubtless never arise. The empirical origin of this conception is also shown by the fact that even the adult mind already schooled in abstract thinking has the greatest difficulty in clearly conceiving it. Perhaps not one in a thousand correctly represents to himself the surface as he has been taught to define it. It is usually imagined as a corporeal sheet of constant thickness which can be made small as we please.

Suppose we consider for a moment the surface of the ink in the bottle before me. This surface, we say, is the boundary which separates the ink from the air that is above it. But what is this boundary? Is it ink? Certainly not, for then we should still need a boundary to separate that ink from the air above it. For the same reason we cannot say that it is air. If, then, we should magnify the two as much as we please and should find that they remain always homogeneous, each filling up the space adjacent to the other, we should still have to say that the surface between them is neither the one nor the other; it is not a layer of air, of ink, of ether, nor of space; it is simply a boundary or geometrical surface, and as such occupies *no space at all.* And yet as we reach this conclusion how difficult it is to avoid falling back upon that class of experiences by abstraction from which this conception was originally obtained, and thus picturing to ourselves an exceedingly thin material partition, or at least some

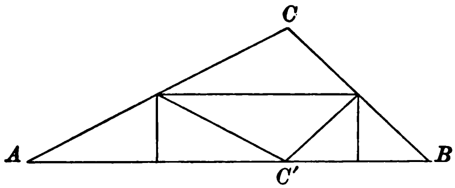
small portion of space which might serve to keep the two substances separate and distinct from each other.

This, of course, does not satisfy reason. To put a space boundary between two adjacent portions of a continuous quantity is not only to regard space as a spread-out, empty reality which, while providing "room" for things and permitting them to approach or recede from each other, nevertheless keeps them asunder; but it is also to propose the old problem over again while doubling its difficulty, for we now want a boundary between the space in question and each portion of the quantity separated by it. It is the same difficulty which is met with in all limiting conceptions. It arises from the disparity always to be felt between actual experience and those intellectual ideals the peculiar character of which this experience itself suggests and determines.

Coming now directly to Euclid's theory of parallel lines, we note that it was also determined to be what it is by a variety of empirical data. If these data had been decidedly different from what they are some Lobatchewsky might ultimately have given us the Euclidean system, but this system would certainly not have been first in the order of development. The data in question are many and simple, and were certainly familiar even to the most ancient civilizations. The ornamental designs of savage tribes in weaving, drawing, wood-carving, and the

like often suggest that the triangle's angle sum is two right angles.

By folding a triangular piece of cloth or paper as shown in the figure, this truth is directly perceived. The angles of the triangle are seen to form a straight angle by bringing their vertices together at C′.[14]



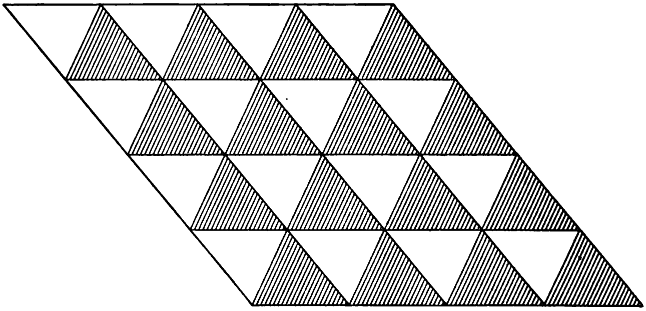
The same truth doubtless suggested itself over and over again to the workmen in clay and stone of Babylonia, Egypt, and Greece in the mosaics and pavements which they are known to have made from differently colored stones of the same shape. In this way, too, it was easily found that the plane "space" about a point can be completely filled only by three kinds of regular polygons, that is, by six equilateral triangles, by four squares and by three regular hexagons,"[15] and hence that this space is always equal to four right angles.

In this ancient paving, triangular stones of the same shape and size were frequently used by placing

[14] Tylor suggests this as the probable origin of this theorem. Anthropology, New York, 1896, p. 320.

[15] Proclus attributes this theorem to the Pythagoreans. Gow, op. cit. foot note p. 143.

their bases on the same straight line as in the accompanying figure.[16]



Here the system of equally distant lines is a striking feature. It is also made clear by this figure that the sum of the interior angles on the same side made by the intersection of any two of these equally distant lines by a third line is two right angles. The obvious fact that this style of paving may be extended without limit leads naturally to the conviction that parallel lines will not meet however far produced, for such lines are seen to be ideally unlimited and yet everywhere equally distant. Thus we have suggested, at least, in this simple figure all the empirical data necessary to the formation of the Euclidean theory of parallel lines.

But by whatever fact or class of facts this theory was originally suggested, the mechanical arts furnish

[16] See E. Mach's valuable article in The Monist, Vol. XII., pp. 481-515.

today innumerable examples of its practical truth. The existence of similar figures of unequal sizes and the actual construction of rectangles whose angles all remain right angles and whose opposite sides continue to be equal when tested by the most accurate measuring instruments are constantly recurring proofs of Euclid's validity. If, however, in the construction of quadrilaterals with angles all right angles, and sides practically the straightest possible, it had uniformly been found that the opposite sides are unequal, as actually happens in surveying such figures of large size upon the surface of the earth, we should then doubtless have reached with equal confidence a different conclusion, and our sciences of mechanics, physics and astronomy would have been quite different from what they are; but as a matter of history this has not been true. From the crudest measurements to the most refined; from the ancient "Harpedonaptai" [17] of Egypt to the present day in the constantly increasing refinement of the powers of accurate observation, man's ability to measure slight deviations from right angles and from the equality of distances has remained relatively equal. There are today no observed phe-

[17] Egyptian "Rope Stretchers" mentioned by Democritus. Their principal duty was the construction of right angles by means of ropes divided into 3, 4, and 5 equal units of length. See Cantor's *Vorlesungen über Geschichte der Mathematik*, Vol. I., p. 64, Leipzig, 1894.

nomena coming within the scope of the physical sciences that seem to contradict in any appreciable degree the Euclidean postulate. Therefore even when we set aside, as we must, Kant's contention for the *a priori* synthetic nature of the parallel postulate, it may still be claimed that there is no law of nature reached by scientific induction that can be said to have so good a right to be called fundamentally and universally true of the world as we know it.

As just stated, the parallel postulate is rooted psychologically, in so far as direct perception is concerned, in the ability to discriminate slight variations in the size of angles, in the length of lines, and in the departure of the latter from "absolute straightness." It requires that this ability shall remain as a rule, relatively speaking, precisely equal in these different directions. Now it is obvious that "the relations between the least perceptible difference of the angles of a parallelogram and the least perceptible differences of the lines forming its sides are exceedingly complex and variable." [18] It is therefore possible that the law of these relations when determined by the most refined experimental analysis [19] will prove to be in *essential* agreement

[18] Ladd: *A Theory of Reality*, New York, 1899, p. 315.

[19] The writer has already begun, at Professor Ladd's suggestion, the experimental problem here referred to. So far, however, the results obtained are not of a very positive character.

with the Euclidean postulate. Indeed this may be even confidently expected in view of the general experiences of the race to which we have referred.

This conclusion, however, is by no means to be regarded as a necessary one. In fact, that ideal exactitude which is required to establish beyond doubt the validity of this postulate is no more to be expected in this than in any other form of empirical testing. The absolute validity of Euclid can never be established by any mere appeal to perceptual experience, however refined. Our space may possibly be proved in this way to be non-Euclidean, but it can never be shown to be exactly Euclidean.

The analysis, then, of the experiences involved in the theory of parallel lines requires the consideration of a special relation between angles and distances, and the problem is to determine the peculiar character of this relation.

We have already seen (Chap. I., pp. 9 and 10) that in defining angles as differences of direction, straight lines, as those which do not change their direction, and parallels, as straight lines having the same direction, we overleap at a bound the difficulties involved in the parallel postulate. It is therefore evident that the peculiar properties of parallel lines are somehow bound up in the word direction, and could be distinguished if the meaning of this word were subjected to a careful analysis. Through the recognition of the importance, for Euclid, of the

idea of direction, Von Cyon [20] was led to believe that, in discovering, as he thought, a special sense-organ in the semicircular canals for the perception of three fundamental directions corresponding to the co-ordinates of the Cartesian system, he had solved the whole space problem completely. But the fact already pointed out (Chap. I., p. 10) that direction as ordinarily understood can only be completely defined when the parallel postulate is already assumed shows quite clearly that such a solution is only apparent. It proceeds, in fact, upon the customary assumption expressed in the above definition that direction is the one essential, sufficient, and peculiar property of the straight line; in other words, that *straightness* and *direction* are exactly equivalent in meaning, and that the latter word is used in precisely the same sense in all three of the above definitions. Now we wish to show that there is unconsciously introduced into this word as it is used in the last two definitions, in addition to the idea of straightness which it always carries, an element of meaning which ultimately arises from the peculiar structure and symmetry of our bodily organism.

It will be remembered that in the early part of the present chapter we called attention to the peculiar symmetrical arrangement of corresponding sensory organs and the influence of this arrangement upon

[20] *Pflueger's Archiv fuer Physiologie*, 1901.

the orientation [21] of the body with reference to any stimulus appealing to the senses. This is no doubt largely due to the remarkable similarity of the sensory experiences of corresponding organs, as, for example, the eyes or the ears. These sensations, however, are not wholly identical, and their differences, combined with other sensation factors, mainly those of movement brought out in repeated acts of adjustment, give rise to characteristic distinctions in sensations of direction. For example, the directions, before and behind, up and down, right and left, as actually experienced, involve sensory differences somewhat analogous to those of color.[22] These differences attach themselves to our notion of direction, and when unconsciously carried over with this notion into the abstract space of mathematics they lead to confusion. There is no such thing as a difference of direction in geometrical space. In such space *two* points are always required to determine

[21] This orientation may even be wholly involuntary, as in the case of the moth when it helplessly flies into the flame and is burned.

[22] It is an interesting fact that Helmholtz was led to investigate the problem of space by the analogy which he perceived between space and the color system as tridimensional manifolds. We shall see when we come to consider the impossibility of carrying to one space the same metrical standard which was employed in another that the so-called space-constants are qualitatively different and yet serially related to each other in a manner which is in all essential respects similar to what we experience in the perception of differences of color.

a line. If a number of lines are conceived of as going out from a point they can only be distinguished by an actually perceived or else an imagined relation to our bodily selves. We can transport ourselves in imagination to the vertex of any angle formed in this way, and by the use of a distinction not inherent in the figure itself we can represent the angle to ourselves as a difference of direction, and thus distinguish the lines from each other. In no other way can we do this; we picture ourselves as it were standing at the vertex of the angle, looking alternately down its sides, and thus in imagination sensing their directions as different.

We carry, then, into our abstract space-world with this notion of direction a misleading subjective distinction. The bodily self enters into spatial relations with the other objects of this world in a peculiar way; it is not allowed to take its place, as this abstract conception of space requires that it should, merely as one among many objects of whose distinguishing peculiarities except in so far as those qualitative similarities are concerned which measurement requires, geometry can take no account whatever.

It is also easy to see that the same "physiological" element of meaning attaches to the word direction as employed in the definition of parallel lines. For what else can be meant by saying that such lines have the same *direction* and that even to infinity? This is clearly apparent if we follow closely the lan-

guage of Mr. J. S. Mill in the following passage:[23] "Though, in order actually to see that two given lines never meet, it would be necessary to follow them to infinity; yet without doing so we may know that if they ever do meet, or if, after diverging from one another, they begin again to approach, this must take place not at an infinite but at a finite distance. Supposing, therefore, such to be the case, we can transport ourselves thither in imagination, and can frame a mental image of the appearance which one or both of the lines must present at that point, which we may rely on as being precisely similar to the reality."[24]

But the mere straightness of two lines lying in the same plane cannot, of itself, justify any statement as to whether they will or will not meet when prolonged without limit. If it could, there would then

[23] *Logic,* Book ii., Chap. V., § 5.

[24] It is interesting to note while this passage is before us that its closing statement is open to criticism. On a preceding page Mr. Mill has stated that "we should not be authorized to substitute observation of the image for observation of the reality, if we had not learnt by long continued experience that the properties of the reality are faithfully represented in the image." Now it is evident that experience can only tell us this in the case of realities and images, both of which have been experienced: both must be known before we have a right to say that the one faithfully represents the other. Hence in admitting, as Mill here does, the universality of the truth of the parallel postulate, he introduces a factor of knowledge which can not, strictly speaking, be gotten from experience alone as he conceives it.

be no need whatever of the parallel postulate, for Euclid becomes established at once when the existence of straight lines in this sense is granted. It is difficult to grasp the truth of this statement, so natural has it become through long-continued association of these ideas to look upon "straightness" and "direction" in their adjective relations as perfectly synonymous terms.

Thus it has come to appear that if only the lines are *actually* straight the parallel postulate must follow of necessity, and that it is only when we have unconsciously or purposely introduced some sort of bending or curvature into the lines themselves that this postulate fails. But careful reflection upon the essential meaning of straight lines when carried back to the source in experience whence this conception has sprung reveals that this *a priori* necessity of the parallel postulate simply does not exist.

Some distinction between the idea of straightness and that of direction must obviously be maintained if any non-Euclidean system is to be taken seriously as having anything to do with reality. If these two words are, in fact, identical in meaning and the word "direction" as used in the definitions of straight lines, angles, and parallels as given above holds precisely the same significance in each definition, it is easy to see that any serious struggle between Euclid and his modern rivals is out of the question, for Euclid alone is left on the field.

A familiar illustration will assist in making this distinction clear. As everyone knows, a curve is frequently defined in elementary geometry as a line which changes its direction at every point. Now if we substitute straightness for direction in this definition it becomes at once absurd. For even if we could give the resulting expression, "changes its straightness," etc., an intelligible meaning by regarding it as signifying the amount of angular deviation from the tangent to the curve at the point considered, the definition would still break down when applied to the circle, for in this case we have a figure whose angular deviation from the tangent line is a constant quantity, and therefore, according to this definition, the circle could not be a curve at all.

We see, then, that the word "direction" names a complex idea which is altogether too inclusive and variable in meaning to specify accurately what is meant by the straight line. It leads to confusion where precision of statement and accuracy of meaning are urgently demanded, and should therefore be avoided if possible.

Generally speaking, the confusion so frequently met with in discussions on the philosophic foundations of geometry arises from the general tendency to regard as simple and without the need or the possibility of further analysis certain ideas which in reality are complex in character and therefore can-

not be regarded as ultimate. Emphasis is laid now upon one phase, now another, of these complex ideas, and the whole problem is solved by overlooking entirely the real question at issue.

Of all the fundamentals of geometry, that which stands most in need of satisfactory analysis is the conception of the straight line. What is there in this word, let us ask, that makes it not only universally applicable, but even indispensable to all forms of geometry? What do we and what ought we to mean by "straightness" as applied to lines in all these systems?

This conception, by virtue of its complication with certain metrical ideas that have always been associated with it, gives philosophically the greatest trouble in any attempted critical estimation of different geometrical systems. Angles present no difficulties of so serious a character. The old Euclidean postulate that all right angles are equal is in reality a theorem which has lately been rigorously proved,[25] and is just as true of non-Euclidean as it is of Euclidean geometry. The angular magnitude about a point is equal to four right angles in any unbounded geometrical surface whose curvature is constant, and is therefore the same in all forms of

[25] Killing, *Grundlagen der Geometrie* (Paderborn, 1898), Vol. II., p. 171; and especially Hilbert, *Grundlagen der Geometrie,* Leipzig, 1899, p. 16.

geometry.[26] Hence the creation of new geometries does not essentially modify the difficulties to be met with in the treatment of angles. Angular magnitudes can always be expressed as ratios of linear magnitudes, and are therefore easily determined when the latter are known. Upon the possibility of such ratios the whole science of trigonometry corresponding to any conception of space is erected.

Let us, then, focus our thought upon the straight line and try by careful analysis to answer our questions. Having already traced the growth of this conception as it presents itself to the consciousness of the ordinary man, let us now turn to the mathematician and endeavor to learn from him what he considers to be essential to the straight line as shown by his definitions. When the results of this analysis have been attained we shall then try to relate them, if possible, to actual experience.

We have seen that Riemann makes room for an unlimited number of geometries differing from each other fundamentally in the meaning assigned to the linear element *ds,* and that in the further working out of his system emphasis was laid upon *curvature*[27] as an important conception. Cayley makes

[26] There may appear to be an exception to this statement in the case of the apex of the cone where this angular magnitude is always less than four right angles, but I have attempted to remove this objection by the word *unbounded.*

[27] We have already noted the development of this concep-

distance fundamental, and with the aid of Klein shows that a variation of the laws by which distance is measured is all that is necessary to distinguish Euclidean and non-Euclidean systems. From this it appears that geometries are to be distinguished by the number and character of the postulates employed to determine certain conceptions which for the ordinary man define the straight line. For him, as we have said, this line is at all times simply the shortest *distance* between two points or else a line which does not change its direction at any of its points. But these words are not simple in meaning, and therefore stand in need of definition themselves. Direction has already been considered and need no longer detain us. Let us now examine the phrase "shortest distance" and endeavor to determine its meaning. Reflection shows that this, too, is far from being a simple conception. It presupposes, in fact, all the assumptions necessary for measurement. It presumes beforehand the possibility of measuring all kinds of lines that can be drawn anywhere in space, else how could it be said that a certain one of these lines is the *shortest distance* between two given points. This is certainly a tremendous assumption. All the postulates demanded by metrical geometry are wrapped up in this definition. Before measure-

tion in the historical treatment of Riemann's geometry in Chapter I.; we shall take it up for more thorough treatment in our last chapter in the discussion of space conceptions.

ment is possible we must have a standard of measurement, and this standard is itself the straight line. Measurement, then, presupposes the straight line as a necessary pre-condition of its own possibility, and therefore cannot be taken as a simpler and more ultimate notion with which to define the straight line. Furthermore, the existence of a minimum is itself an assumption which involves certain logical consequences of an interesting character, which cannot be dealt with here. Finally, this definition fails in the case of straight lines which join antipodal points of double elliptic space. Between such points there is an infinity of shortest lines.

These are difficulties which are certainly of a serious character. Nevertheless it seems hardly possible wholly to avoid some conception of distance when we talk of straight lines. Long ago Leibnitz made distance fundamental,[28] and the same point of starting has recently been taken by Frischauf and Peano. Peano defines the straight line *ab* as a class of points *x*, such that any point *y*, whose distances from *a* and *b* are respectively equal to the distances of *x* from *a* and *b*, must be coincident with *x*. But Peano fails to prove either that such a line exists or that, if it does exist, it is determined by any two of its points.[29] This, of course, is impossible without the use of certain special axioms. According to

[28] Russell, *Principles of Mathematics,* Vol. I., p. 410.
[29] Russell, *Principles of Mathematics,* Vol. I., p. 410 ff.

Peano, five such axioms are needed. The group which he has selected is an interesting one, because it defines distance by the use of *between* as an indefinable notion. His axioms for the straight line *ab* are as follows: (1) Points *between* which and *b* the point *a* lies; (2) the point *a*; (3) points *between* *a* and *b*; (4) the point *b*; (5) points *between* which and *a* the point *b* lies.[30] Just what is meant by *between* is nowhere clearly explained.

Vailati attempts an explanation which is rejected by Peano[31] on the ground that *between* is a relation of three points and not of two only. As a matter of fact if we confine ourselves to projective geometry even three points on a line are not so related that any one of them can be said to be *between* the other two. The word, between, involves a certain ordering of the points which in projective geometry depends upon the nature of the quadrilateral construction which requires for its proof a point outside its own plane and hence is not possible without three dimensions. It also requires four perspective triangles. The generation of order by this method is therefore considerably complicated and the simplest proposition involving *between* which remains unaltered by projection is one which requires *four* points.

It is obvious then that if "between" is to be established at all as a unique relation of any *two* points

[30] *Rivista di Matematica,* Vol. IV., p. 62.
[31] *Rivista di Matematica,* Vol. I., p. 393.

it cannot be done by any appeal to projective considerations. But in spite of this Russell[32] sets Peano's criticism aside as inadequate and adopts what is practically Vailati's position. He avoids the word "between," however, and introduces in its stead a new indefinable. He posits a class of *asymmetrical transitive relations*[33] which he calls K and assumes that between any two points there is one and only one relation of this class. Eight axioms which Russell shows to be distinct and mutually independent are required, as he thinks, to define what is meant by this class of relations.[34]

Avoiding his symbolism these axioms may be stated as follows: (1) There is a class of asymmetrical transitive relations K; (2) there is at least one point, and if R be any term of K we have; (3) R is an aliorelative, that is, a relation which no term has to itself; (4) the converse of R is a term of K; (5) $R^2 = R$, (6) the domain of the converse of R

[32] *Principles of Mathematics*, Vol. I., pp. 394 ff.

[33] In the sense in which these terms are employed by Mr. Russell they may be explained as follows: When the converse of a relation is the same as the relation itself, the relation is said to be symmetrical; but when the converse and the original relation are incompatible the relation is said to be asymmetrical. Examples of the former are such as *identity, equality,* and *inequality;* and of the latter such as *better* and *worse, greater* and *less.* A relation is transitive when any such condition as the following holds: viz., if *a* be similar to *b* and *b* similar to *c,* then *a* is similar to *c.*

[34] *Princip. Math.*, Vol. I., pp. 395-396.

is contained in the domain of R; (7) between any two points there is one and only one relation of the class K; (8) if *a, b* be points of the domain of R, then either *a* holds the relation R to *b* or *b* holds this relation to *a*. The seventh axiom is obviously double. It asserts (1) that there is one such relation between any two points, and (2) that there is only one. The first member of the group is not an axiom at all but only the assumption of the indefinable class of relations K.

With this outfit of assumptions the various forms of geometry may be constructed. The mutual independence of the entire group is shown in the usual way. Any one of them may be denied and a logically consistent system built upon what remains. The fifth and seventh axioms are the most interesting here. In (5) we may deny either that R is contained in R^2 or that R^2 is contained in R. If we deny the former the resulting straight line becomes a series of points which does not possess the "density" or "compactness" which the constructions of Euclid and non-Euclid require. The general result, however, is logically sound, the series in question simply lacks the degree of continuity which geometry requires. If we deny the latter, the result proves to be untrue of angles which otherwise may be made to satisfy all the conditions expressed by this group of axioms. If a Euclidean and a hyperbolic space be considered together all the axioms

still hold except the first part of (7). The whole group with the exception of (7) is shown to be necessary, but for (7) Mr. Russell is only able to maintain a high degree of probability.

We have now to consider another interesting effort to reach the "minimum essential" to the notion of "straightness." This lays the emphasis upon a different factor of experience from those just considered, and consequently appeals very strongly to those who prefer to approach geometry from the motor side, rather than from that statical conception of an "empty" space which may be reached by abstraction from the materials furnished by vision alone without reference to bodily movements actual or imagined.

The effort to which we refer is that of Pieri,[35] who proceeds to build geometry upon the two indefinables, *point* and *motion*. He defines the straight line joining two given points as the class of points whose internal relations remain unchanged by any motion which leaves the two points fixed. His system is simple and logically unimpeachable.[36] But here again we are dealing with a complex idea. Motion as used by Pieri is not simply the motion of a single point, but a certain law of motion is assumed along with this simpler idea. We have the motion

[35] *Della geometria elementare como sistema ipotitico deduttivo*, Turin, 1899.

[36] Russell, *Principles of Math.*, p. 410.

of a number of points taking place in such a way as to leave certain relations unchanged. In other words it is the motion of a rigid body which is here assumed.

If then we ask how motion as thus understood is to be distinguished from other transformations it becomes readily apparent why Pieri was able to construct metrical geometry by considering such a notion as indefinable. For when we attempt to answer this question we see that in this idea of motion all the conditions necessary to metrical geometry are already assumed. Pieri means simply a transformation which leaves all essential metrical properties and conditions unchanged. Having thus presupposed metrical properties in his conception of motion, he then turns about and defines these same properties in terms of this conception. Little wonder that he succeeds.

It must be observed also that Riemann's famous dissertation involves the same difficulty. He assumed that the linear element ds remains unaltered by the same infinitesimal motion of all of its points, and bases his system upon this assumption. He thus presupposes the existence of equal spatial quantities in different places which is equivalent to the assumption of spatial homogeneity, or the free mobility of rigid bodies. In other words by assuming metrical properties in his ds and then proceeding to determine these properties upon the basis of this

assumption, he easily draws out at the faucet what he has already poured in at the bung.

Glancing back now over the definitions which we have just considered in the light of the critical examination to which we have subjected them let us briefly summarize the resulting facts. We have found that in each case certain metrical properties at some stage or other in the process were either unconsciously assumed or else openly postulated. Peano starts with distance and closes his discussion by grounding this conception upon "between" as an indefinable notion. Russell, though holding essentially the same meaning, dispenses with this word and substitutes in its place a particular class of asymmetrical transitive relations by means of which the order of points on the straight line may be generated in a perfectly definite way. And finally we found Pieri employing in his definition the idea of motion which he assumed to take place in such a way as to allow the unique character of the straight line as well as certain metrical properties to remain wholly unchanged.

If now we sever these words from their abstract logical surroundings and endeavor to carry them back to a meaning in experience, we find that each word in its turn bears with it what is formally essential to the idea of the straight line. All the way through, this conception of the line as reached by abstraction from certain forms of experience already

pointed out, and defined as the shortest distance between two points, or as a geometrical object which when rotated about any two of its points remains always within itself, has really been present determining the result. It is this conception undoubtedly which has guided Peano and Russell in selecting the postulates by which their conceptions are at last defined. It is this conception and naught else which has determined the number and character of these postulates to be what they are. There are certainly no *a priori* reasons which independently of those actual experiences in which the notion of the straight line is ultimately grounded, could have decided this matter. Why select this peculiar conception at all as the one most convenient and therefore best suited to serve as *the* point of starting in all systems of geometry, and why ascribe to it just those peculiar marks that we do? It takes experience to answer this question. No amount of mere passive contemplation of blank space, if such a thing were possible, could ever lead to a direct intuition of the peculiar properties of the straight line. We can reproduce such an object in imagination, it is true, but in reality this is but an "experiment in thought" which could not take place except upon a basis of past experience.

We do not wish to be misunderstood. What we are here contending for is the empirical origin of the special properties of the straight line which dis-

tinguish it as a peculiar relation between two points. We do not mean to assert that there is nothing *a priori* about this conception. The straight line is a peculiar relation between points. It both separates and unites these points in a relational way. The points can not coalesce, and become one point, they must be kept distinct from each other, else the line which they are supposed to determine vanishes away. So much is true of every straight line however its peculiar characteristics may otherwise be determined, whether we represent it to ourselves by means of a visual, a tactual, or a motor image. So much at least appears to be involved in the very notion of space as an externalizing principle which renders possible any system of distinct coexisting entities. Whatever is more than this has a basis in experience, and whenever we conceive such a line to be something which admits of definition it becomes much more than a mere abstract relation between points, it becomes a *straight* relation, the question of the economy of effort represented as being somehow felt in passing from one point to the other now enters into the conception and carries us back at once to data which we can only represent to ourselves as being furnished by some form of sensory experience, real or imagined. How shall I proceed in the shortest time possible and with a minimum of effort from one point to the other? That pathway if you please which answers this question is

the straight line. If we now ask just how this pathway is to be determined, we fall back upon certain special experiences such as we have already described. The line which best satisfies this requirement of a minimum of effort for us *tridimensional beings,* is one which is congruent with itself, in all positions, which is the shortest distance between two points which rotates within itself, which looks the same from all points not in it, or one whose internal relations remain unchanged by any motion which leaves two of its points fixed. If we neglect the question as to the number of space dimensions which are to be taken account of, these definitions all mean essentially the same thing, but they emphasize somewhat different aspects of the complex experience involved.

This conception then is a generalization from facts which come within the limits of experience and must therefore fit these facts when carried back to them. It is only when we go beyond the facts and, by following the lead of suggestions offered by experience or by investigating the logical possibilities involved, pronounce upon such matters as the infinity or two sidedness of the straight line that the roads leading to Euclid and non-Euclid begin to diverge. This may be made clear by the following illustration. Suppose we have two perfectly straight lines lying in any position in the same plane and extending without limit in both directions; let

one of these lines rotate about a given point not on the other line, starting the rotation with the lines intersecting. The point of intersection moves along the fixed line either away from or toward the fixed point of rotation according to the direction in which the line is rotated. For convenience let it move away from this point. Three different results are logically possible. When the rotating line ceases to intersect the fixed line in one direction it will immediately intersect in the opposite direction,[37] or it will continue to rotate for a time before intersection takes place, or else there will be a period of time during which the two lines intersect in both directions. The first of these possibilities gives Euclid's, the second Lobatchewsky's, and the third Riemann's geometry.

The mind's attitude toward these three possibilities taken successively illustrates in a curious way the essentially empirical nature of the straight line as we conceive it. Logically one of these possibilities is just as acceptable as the other. From this point of view strictly taken there is certainly no reason for preferring one of them to another. Psychologically, however, Riemann's hypothesis seems absolutely contradictory, and even Euclid's is not quite so acceptable as that of Lobatchewsky. The lines cannot intersect in both directions, we say, for

[37] We can not say the *opposite end* for the lines in each case and in both directions are supposed to be unbounded.

in that event they would necessarily be curved or at any rate not continuous straight lines. Even the assumption which leads to Euclid seems not to be in accord with the lines of experience which we call parallel; for all such lines must turn a little way after ceasing to intersect at one end before they begin to intersect at the other. The logical requirements which in this case lead to Euclid, seem to be at variance with those very experiences which have produced this geometry. The lines seem to be as pictured by the imagination, necessarily curved. Suppose that the point about which the rotating line turns is at a great distance from the fixed line, it then appears wholly impossible to our picturate thinking to represent the two lines as straight under the given conditions. They appear to be geodesics joining antipodal points on the surface of an immense sphere. The reason for this appearance is obvious. It is because of our familiarity with finite spherical surfaces whose straightest lines are curved and are known to intersect in the manner assumed by Riemann. Hence in harmony with the mind's general tendency to settle its abstract ideas upon some convenient concrete model perceived or imagined, Riemann's straight lines become at once attached to the sphere and are thus pictured as curved. In Lobatchewsky's assumption the case is quite different, for it seems to contradict none of our ideas of the straight line. Beltrami's pseudo-

spherical surfaces upon which all the theorems of Lobatchewsky may be projected and there visualized as figures whose sides are curved, are not at all familiar to us. But suppose these saddled shaped surfaces had been the ones with which our notion of curved lines had always been associated and that pseudo-spheres and spheres had simply exchanged places in our experience, would not Lobatchewsky's assumption in that event have appeared to be just as incompatible with our notion of straight lines as Riemann's does now?

Again there is logically no reason why the geometry of Lobatchewsky should have appeared in history before that of Riemann. We have seen, however, that the former assumption worried Saccheri very profoundly in his heroic efforts to "vindicate Euclid," more than a century and a quarter prior to the publication of Riemann's dissertation. It was not a logical but a psychological or experiential difficulty which caused Saccheri to reject the logical conclusions to which his own labors clearly and inevitably pointed; and it was certainly the same sort of difficulty which caused the immediate rejection, by himself and by subsequent mathematicians, of the assumption upon which Riemann's geometry is grounded.

We must now briefly summarize the results of the present chapter and indicate the direction which in

consequence of the positions here taken, our further treatment of the problem must follow.

We have endeavored to show that all strictly metrical conceptions, including the straight line and the theory of parallels, grow out of experience with the objects of our environment, and that the particular form which these conceptions have assumed is determined to be what it is by the peculiar character of this experience. This is true not only of the special conceptions by which the different forms of geometry are defined, but it is also true of the space conceptions which lie at the foundation of each. Geometrical spaces are abstract ideal conceptions which arise out of experience in much the same way and by means of essentially the same faculties as those which are employed in the development of other conceptions. There is nothing especially mysterious about them. They differ fundamentally as to the logical possibilities which the same general experience suggests in regard to what may conceivably happen beyond its actual realm, and also as to the particular aspects of this experience upon which emphasis is especially laid. From somewhat heterogeneous psychological spaces through the physical and geometrical on up to that higher conception which in a sense must include and harmonize them all, the process is a unitary one and essentially unbroken. By fusing together the spatial data derived from the various senses

through ignoring the differences between their several deliverances and by correcting the appearances to one sense through those of another, in such a manner as to give the most complete and trustworthy perception of the objects of our environment in their relations to ourselves and to each other, we derive our notion of space in which the physical world is apparently set. We simply manipulate the data of sense as if always with an end in view, namely that of perceiving the things of our physical world successfully and adjusting ourselves comfortably to them. At a higher stage the same purposive process yields those adjectives which define the various forms of geometrical space. Just as the varying appearances of things to the different senses for example were ignored in order to arrive at their real place in our mental picture of the world, so the varying and irregular deformities to which they are actually subjected in different places by virtue of their relations to each other, when abstracted from lead directly to the conception of relativity and homogeneity of space. In a similar fashion notions of unboundedness and infinite divisibility arise.

We have pointed out the indispensableness of the straight line and indicated its relation to the parallel postulate. The two are bound together by no logical necessity but by a relation which is *assumed* to hold in a region which lies beyond all experiences actual or possible. This conception of the straight

line when freed from certain ideas which, by contingent, though long continued and practically uninterrupted associations have served to disguise its essential meaning, is, so far as it is grounded in actual experience, the most fundamental figure in all forms of geometry. Straight lines are straight lines in non-Euclid as well as in Euclid and although we may, to aid the imagination and for convenience of study, conceive the figures of these newer geometries to be projected upon surfaces in finite portions of Euclidean spaces of higher dimensions — surfaces whose lines are all curved; yet we must not confuse these props of the imagination with the realities for which these geometries as logical systems may be supposed to stand.

When we go beyond the experiences which have given us this conception of the straight line, as in thought we may, and consider the different logical possibilities which lie open to us there, we get by postulating these possibilities in succession in conjunction always of course with those idealizations which actual experience affords, the different systems of geometry.

There is then, and must always remain a gap between empirical and mathematical exactness, between experiences actual and the ideals which experience suggests. In the case before us there is such a gap between the conditions imposed by sense-perception and the mathematical precision which

results when these conditions are supposed to be withdrawn. In this opening, as we have said, logical possibilities lie which are seized upon and woven into systems by the different geometries. Under the growing refinement of methods and means of observation this opening is a decreasing variable; it is therefore not only conceivable but empirically possible that an appreciable deviation of the parallel postulate from the space of experience may eventually be found. In other words, as tested by the court of perceptual experience alone the case may some day go against Euclid but it can never be decided in this way absolutely in his favor.

We reach then at the close of this chapter the important conclusion that Euclid's validity must be established, if established absolutely, upon some other basis than that of mere perceptual experience. Whether such a basis exists remains to be seen.

THE NATURE AND VALIDITY

OF THE

PARALLEL POSTULATE.

CHAPTER V.

THE NATURE AND VALIDITY OF THE PARALLEL POSTULATE.

We have reached the conclusion that geometrical spaces, merely as such, are all of them abstract conceptions. They are grounded on and grow out of the same general experience which they interpret differently while seeking to simplify and to systematize it by means of the peculiar postulates which define them. We can also see that the quantitative concepts which underlie the different geometries have been chosen somewhat arbitrarily so that when we carry them back to the facts of spatial experience they do not reproduce these facts in any case with absolute precision. Different groups of ideas may therefore serve to express all the facts with equal exactitude within the region accessible to observation.

The space-world as we *know* it is not quantitatively infinite nor does the assumption that it is seem to be a necessary one, although Euclid requires it. If then we refuse to call it infinite in this sense but still accept the present laws of optics and astron-

omy which presuppose Euclid's validity and if we also admit the postulate of free mobility which is the same as to assume that the parameter of space (K)[1] is a constant quantity, we find that even under these restrictions it is still possible to represent all the facts with equal accuracy by the geometries of Euclid, Lobatchewsky, and Riemann. So far then as experience goes at present, or can ever go for that matter, there is no necessary reason for starting any physical inquiry with the Euclidean assumption that K is infinite. All we need is to take K sufficiently large to make the deviation from Euclid fall within the limits of astronomical observation. As this observation becomes more refined and exact, one of two things must inevitably happen. Either the facts will ultimately appear to go against Euclid or else it will be shown that the actually *known* quantity of space, though always finite, will transcend successively certain increasingly large amounts which the new approximations to the value of K will give us the right to affirm.

But to go so far as to assert even that K is constant seems, on the surface at least, to be an arbitrary matter which is not demanded either by experience or by logic. It means the same thing as to assert an absolutely rigid standard which may be transported unchanged to any part of space.

[1] See Chapter II. under the discussion of Riemann's idea of curvature.

But we certainly do not meet with any such rigid measures among the objects of experience from which as we have seen the idea of a metrical standard has actually been derived. What we really know is that some of these objects are less subject to quantitative variations than others are, and that consequently a series of them may be arranged whose members as we ascend the scale approximate more and more nearly a quantitative invariability. Hence here as in the case of the straight line the intellect may form the pure abstract conception of a rigid body which, when thought of as independent of position, furnishes also the conception of spatial homogeneity. But between the actual facts of experience and the essential properties of such an intellectual construct there exists, as stated, a gulf which sense-perception of itself cannot bridge. It is then both logically and empirically permissible to assume that the actual parameter of space is a variable quantity which oscillates within certain narrow limits.[2] By assuming this oscillation to occur in accordance with law other geometries compatible with experience could also be obtained.

It is such considerations as these that force upon

[2] When we assume the possibility of measurement in the exact sense required by geometry, and the facts of experience make this of course a legitimate inference, this assumed variability of K must be ruled out on purely logical grounds, as we shall later attempt to show.

us the necessity of making and maintaining a careful distinction between the *facts* of experience and those *intellectual constructs* whose formation these facts have suggested. Upon these *constructs* two conditions are imposed: they must fit the facts to which they relate, and must also meet the logical requirements of mutual non-contradiction. When these conditions are fulfilled, that is, when it is shown that different systems of geometry, Euclidean and non-Euclidean, are equally permissible under both requirements, the question of validity assumes its most interesting and difficult form.

We have shown that these geometries within certain limitations already pointed out, are empirically indistinguishable; it therefore remains to consider the other half of the problem. Are these geometries when considered strictly from the logical point of view equally tenable? Fortunately, as our historical chapter has abundantly shown, we already have all that seems to be desired for a satisfactory answer. Granting the groups of assumptions from which they set out, accepting the condition merely that these assumptions be so defined as to be mutually independent and logically consistent, and finally, disregarding the question as to their easy compatibility with the known facts of reality; almost an unlimited number of geometries can be, and very many indeed have been, actually built up

in such a manner as to satisfy the strictest demands for internal consistency.

When we confine ourselves then to those considerations which the idea of exact measurement requires, and hence to those assumptions only which are in all respects very similar to Euclid's, the demonstrated fact that the proofs of the resulting non-Euclidean systems hold good for corresponding theorems of Euclid's geometry when the appropriate substitutions have been made, shows beyond question that any logical defects which may possibly hereafter be discovered in any non-Euclidean systems must also apply with equal force against Euclid. A revelation of contradiction in one must prove to be a revelation of contradiction in the other.

Therefore the apparently inevitable conclusion which we are forced to face is that Euclid is neither empirically nor even logically necessary to the world of reality. It is based upon a space *conception* derived by abstraction from that world as we know it and all its theorems and constructions are consequently in perfect logical harmony with that conception. But there are other conceptions derived in essentially the same way from the same world and defined by slightly different postulates. From these conceptions also other geometries are known to flow with equal harmony and necessity. What one then of these opposing systems is the true geometry? It seems almost absurd to ask this question.

They are all, of course, true — true logically and, within limits, true to the facts; therefore the decision between them, if made at all, must be made not upon a basis of truth but simply as a matter of *convenience.* It is not a question of necessity but one of *utility.* When therefore it is generally admitted that Euclid is the most convenient of them all the obvious conclusion is that Euclid will continue in favor though robbed of that peculiar majesty as a system of absolute truth which it formerly seemed to possess.

But we cannot dispose of the difficulty in this easy fashion. Again we have solved the problem by simply hiding its meaning. The word convenient here introduced is employed by M. Poincaré as though it ended the matter. The axioms of geometry are for him all of them mere conventions, they are all true, but some groups are simply more *convenient* than others; but in reality the central philosophical puzzle originating in metageometry lies concealed in Poincaré's use of this word. We cannot dismiss the matter on the mere ground of *convenience.* Granted that Euclid is the most convenient, how, why, and in what sense is it so? We certainly cannot say that it is because of any observed matters of fact to which geometry relates, nor of any requirement of logical consistency. Nor can we say that Euclid is logically the most simple,

for from this point of view the single elliptic system is much more beautiful and attractive.

It is true that from the standpoint of algebra the constructions of Euclid do not involve the use of such complicated equations as are required by his rivals. For these constructions nothing more difficult is involved than a general equation of the quadratic form. Hence the Euclidean lines and planes do not require so high a degree of continuity as that which is demanded by Lobatchewsky and Riemann. The Euclidean plane may be regarded in fact as a continuous two-dimensional manifold of points, some of which have been dropped out at regular intervals. All the constructions of Euclid can be made in such a manifold without anywhere falling into a hole. To sense, it would appear as a sieve, and the straight line as a picket fence where the pickets and the distances between them correspond to the points on the line and their relations to each other.[3] This sort of simplicity, however, results from the application of numbers to Euclidean conceptions and can hardly be regarded as necessarily inherent in these conceptions themselves.[4] To

[3] Compare Dedekind: *Was sind und was sollen die Zahlen*. Brunswick, 1893. *Vorwort s.* XII.

[4] Professor G. B. Halsted states in a letter to the present writer under date of Jan. 4, 1904: "I build up my Rational Geometry wholly without number or arithmetic, using no ratios, irrationals, or complexes, as you will see when it appears in a month or two."

say that Euclid is the most convenient in this sense is but to say that his conceptions happen to be most amenable to easy algebraic interpretation. But this is certainly no mere accident. Like every other fact of human experience it demands philosophic recognition. Why indeed should it be that the very conception of space which is historically first and at the same time most natural, should also be the most easily interpretable by means of algebraic formulæ which have been derived from a widely different and apparently independent source in experience? Why does that which is spatially the most simple and which can be successfully handled without regard to number, thus prove to be at the same time most convenient for numerical interpretation?

In attempting to answer this question the influence of custom should not be ignored. Relations which have been established by long and uninterrupted associations are not easily distinguished from those which are logically necessary. Hence it must not be forgotten that civilization has been steeped in Euclid for more than two thousand years so that today this geometry underlies all our physical sciences and we are giving expression to Euclidean forms in all the mechanical arts.

During this long period the reasoning and even the form of Euclid have been generally regarded as the most perfect model of scientific and even of philo-

sophic[5] thought and expression. Our intellectual life has become attuned both to Euclid's doctrine of space and to his peculiar form of expression. Little wonder then that it seems almost sacrilege to depart from them or even to call them in question; we come to feel with Kant and with thousands of others, that if there is an *a priori* necessity anywhere in human knowledge we find it in Euclid. But suppose all this had not been, and there is surely nothing violent in this supposition, could we still say that Euclid is the most convenient geometry?

We have seen how the peculiar facts which have pointed unmistakably to Euclid have been the most patent, universal, and familiar at all stages of the race's development. They have not needed prolonged meditation, observation, and experiment to bring them into prominence, they lie open and everywhere ready at hand. In consequence, Euclid has gained a long start in advance of his competitors in the race for general acceptance, and popular favor.

But here again we must face the question which seems to confront us withersoever we turn. Why

[5] Spinoza's Ethics is a good illustration of Euclid's influence in this direction. Even the fundamental ontological conceptions have often been borrowed directly or indirectly from Euclid. This veneration of Euclid as a body of ideal knowledge has proved exceedingly mischievous in many ways. Mathematics and formal logic instead of being the very ideal of truth are in an important sense the farthest removed from truth.

this ancient and more favorable start? It is certainly not a mere matter of chance that our world of experience should be so favorably disposed for the suggestion of Euclid. Though the mere antiquity of this science, in itself, is no proof of its absolute and exclusive necessity; no sure demonstration that it, as against other logically and empirically justifiable possibilities, is alone *a priori* and native to the constitution of the mind; nevertheless it does, when taken in conjunction with the matter of algebraic simplicity already pointed out, compel us to consider Kant's doctrine. There is certainly something, either in the knowing mind or the world that is known, which makes for Euclid, and philosophy is called upon to locate it and to determine satisfactorily its essential nature.

Is Euclid then, as Kant maintained, based on an *a priori* form of our sense intuition? *Must* we see things whether we will or no, through Euclidean glasses? And are the non-Euclidean systems after all merely ingenious and interesting intellectual constructs which cannot even be thought of as realized except in Euclidean spaces?

The first of these questions, in the light of our previous discussion, is now comparatively easy to answer; the last will be reserved for the following chapter. At present then we shall have nothing to do with those spatial characteristics which Euclid and non-Euclid hold in common with each other.

We are concerned merely with that peculiar feature which distinguishes Euclid. Dehn's investigation has put beyond question what Euclid's one essential peculiarity is; our present task therefore concerns merely the parallel postulate. Can it be maintained that this postulate is *a priori* in Kant's meaning of that term? If it cannot, then Euclid's last claim of necessary supremacy must be rejected.

By space Kant meant, of course, a universal and necessary form of sense-intuition which is native to the mind. By establishing this thesis as a firm ground of standing he sought to explain how it is possible to have a knowledge of objects not only prior to all experience with them but which transcends the very bounds of all possible experience.

Kant never raises the question as to whether *a priori* synthetic knowledge of any sort is possible. "Of course it is possible," he would say, "in fact, it is actual, for we have it already in Euclid." Indeed Euclid in Kant's day was, and had been for centuries, universally accepted. Hence to prove beyond cavil that certain knowledge of the real world was possible independent of experience Kant had but to point to Euclid. There geometry as a beautiful science stood, that it had apodeictic certainty could not be doubted, for "none but a fool could doubt its validity or deny its objective reference." An *a priori* synthetic body of knowledge is therefore possible; but how is it possible? That was Kant's

problem. If, as he thought, Euclid has apodeictic certainty then space must be *a priori* and purely subjective; conversely, if space is subjective geometry must have apodeictic certainty. Hence his argument assumes a twofold form. On the one hand geometry exists as a science and is known to have apodeictic certainty; therefore it follows that space is *a priori* and subjective. On the other hand, it follows from considerations which are independent of geometry that space is subjective and *a priori,* therefore geometry must have apodeictic certainty.

We now purpose to show that, in the light of our previous discussion, the first of these two arguments, taken by itself, falls short of the mark, and makes against Euclid; and that the second is just as valid for non-Euclidean space-forms as it is for Euclid's and applies in so far as it can be accepted at all only to what these space-forms possess in common. The position which we shall thus aim to establish is that the parallel postulate is not *a priori* but empirical in character.

Kant's first argument, in so far as it is distinct from the second, infers from the mere existence of Euclid the *a priori* and subjective character of Euclidean space. Prior to the introduction of non-Euclidean geometry this argument seemed of course to be a forcible one, but now the case is quite different. For if we do not beg the whole question at the outset by dogmatically asserting that Euclid has

intuitive certainty while non-Euclid has not, and if we also accept the position already established that these geometries are otherwise on a par with each other, it must be admitted that Kant's argument applies as well to the modern systems as it does to Euclid. But when the argument is thus extended its invalidity becomes at once apparent. For if we assume three *a priori* space forms corresponding to these three geometries, and we dare not now make this assumption with regard to one without also making it with regard to the others, they enter into a hopeless conflict with each other. For if it must be universally and necessarily true that the sum of the angles of a triangle is exactly equal to two right angles it cannot also be, at the same time, universally and necessarily true that this same sum is also greater and less than two right angles according as we happen to be speaking of Euclidean, Lobatchewskian, or Riemannian space.

The mere existence of Euclidean geometry is therefore not in itself sufficient to prove the possibility of an *a priori* intuition of Euclidean space. Logically this is a distinction which can no more be claimed for Euclidean than for non-Euclidean geometry. We now see that it cannot be claimed for both without contradiction, therefore it cannot be claimed for either, and this division of Kant's argument falls to the ground.

But in proving the invalidity of this one argu-

ment we have not done with Kant. In the foregoing refutation we have made use of the principle of contradiction to establish our position; but Kant denies that this principle applies. He claims that we can frame an intuition of Euclidean space and that *a priori*. On page 20, for instance, of Müller's Translation of the Critique of Pure Reason, Kant affirms that all geometrical principles as, for example, "that in every triangle two sides are together greater than the third side," *are never to be derived from general concepts* of side and triangle but from an intuition, and that *a priori*, with apodeictic certainty. Kant here sets up the claim that geometrical reasoning is not fundamentally a mere matter of logical consistency, but by virtue of our supposed intuition of space, it is synthetic and cannot, though *a priori*, be upheld by the principle of contradiction alone.

Admitting the soundness of this claim for the present, let us examine the arguments which Kant advances to support it. These are embodied in his general doctrine of the *a priori* synthetic nature of geometrical judgments and also in the five arguments given on pages 18-20 of Müller's translation. The latter may be omitted here with the simple remark that if admitted at their face value they prove nothing peculiar to the special nature of Euclidean space. When taken into full confidence they only establish the necessity of space as a form

of externality, a sort of differentiating principle whose special nature is left undetermined. For our purpose then we need only consider the argument from the *a priori* synthetic nature of geometrical judgments. Kant held that these judgments are not deducible from logic; their contradictories are not self-contradictory. They combine subjects with predicates which cannot be shown by logic to have any connection whatever, and yet these judgments have apodeictic certainty. They are apodeictic not simply because we have a subjective conviction that they are, but because without them experience itself would be impossible. Let us examine this claim.

Since Kant's day attention has been called to the fact that any judgment may be either analytic or synthetic according to the point of view from which we consider it.[6] While, therefore, this distinction so sharply drawn and so greatly emphasized by Kant has a logical value as marking predominant aspects rather than exclusive characters in the classification of judgments it is really without value for a theory of knowledge. For if every judgment is analytic in one aspect, at least, the principle of contradiction may be applied to all judgments without exception, and our previous argument holds.

To make our meaning clear let us take Kant's own example of a synthetic judgment: " All bodies

[6] Consult Bosanquet's Logic, Vol. I., pp. 97 to 103.

are heavy." This judgment he claims is purely synthetic, but is it not also analytic? If the subject signifies one's developed, every-day notion of body, the judgment is obviously analytic, for weight is as much an inseparable attribute of this notion of body as is extension itself. If then this judgment is to be regarded as synthetic we must go back to the origin of the concept body and ask how it was that the attribute weight came to be joined to it. If one had no muscular sense, which alone acquaints one with facts of resistance, his conception of body would never involve the attribute weight; but if he came somehow to be suddenly possessed of this sense and through actual experience with bodies became conscious of weight, the judgment, "All bodies are heavy," would for him be synthetic, for it adds a new predicate to his concept body.

But on the other hand let us suppose a being born into the world in possession of all the special senses except sight and touch. For such an hypothetical being the judgment "all bodies are heavy" would certainly be analytic for weight must be for him an inseparable attribute of body. If now we restore the lost senses of sight and touch and allow him to see and handle objects until through experience he is able to form the judgment which Kant calls analytic, "All bodies are extended," it is plain that for him such a judgment would appear to be syn-

thetic, for as before it adds in the predicate something not previously found to be in the subject.

But it may be objected that Kant's distinction between analytic and synthetic judgments was not based upon the origin, but upon the nature, of the connection between subject and predicate. In reply we need to discriminate between the essential nature of the judgment and its symbolic expression. Failure to distinguish the judgment itself from one's mode of apprehending it often occasions the error of regarding as purely synthetic a judgment which is in fact analytic. If we but attend to the nature of the judgment itself regardless of its external expression and how we came to know it, the relation between subject and predicate is seen to be an identical one. This we believe to be true of geometrical judgments in general;[7] the principle of identity rules throughout. This becomes evident when we consider the way in which such judgments are actually extended. This extension does not involve a necessary synthesis; nor does it depend upon the universality and necessity of space as an *a priori* form of sense-intuition, but rather upon the identity and homogeneity of space as an abstract conception. It is therefore analytic and follows as a natural consequence from this homogeneous nature of space. All the judgments of physics are *hypo-*

[7] Consult Ladd's *A Theory of Reality*, pp. 331 ff.

thetically a priori, that is to say, the connection between subject and predicate is strictly necessary, provided we may be allowed to assume that the subject remains always identically the same. But in this science the subject as a rule does not meet this requirement, for as a result of further investigations it is liable to change. In space, however, we have something which is homogeneous and identical always and throughout; so that in it, because of this peculiar character, such a thing as difference or change cannot occur. Consequently all geometrical judgments, which do but exploit the nature of space, flow directly from this concept by the principle of identity. But this identity or homogeneity of space as we have already shown is not an intuition but a conceptual property. Geometrical judgments are dependent on the nature of the space concept, and simply define its content. These judgments are therefore analytic in nature and consequently, here as elsewhere, the principle of contradiction may be applied.

Euclid's judgments, therefore, are not something *sui generis,* nor is this particular space-form proved by Kant to be an intuitive necessity presupposed in the very possibility of sense-experience; nor is Euclidean reasoning possessed of any peculiar certainty that can be denied to non-Euclidean systems. Geometrical certainty of any sort, Euclidean or non-Euclidean, is but the certainty with which any con-

clusions follow from non-contradictory premises. It is logical certainty only, and in each case flows directly from definition. The certainty with which the sum of the angles of any triangle may be asserted to equal two right angles in Euclidean geometry is exactly the same sort of certainty as that by which it may be shown to be more or less than two right angles in the other two geometries. In each case it is but the certainty of intrinsic consistency.

Kant has certainly failed to prove the *a priori* necessity of Euclidean space. Therefore the objection often raised that we have no intuition of non-Euclidean space-forms comes without force against these systems until some one has shown that Euclid is in any special sense intuitively certain.

Our conclusion therefore is that the parallel postulate is not a subjective necessity; it is not essential to the constitution of man's intellectual nature in the strict sense in which Kant claimed it to be so. There remains nothing, so far as we can see, on which such a claim may be grounded except the mere subjective conviction that it *must* be so; obviously this conviction cannot count for much in the case of one who simply does not feel it. Furthermore any such extreme doctrine of the exclusive subjectivity of space defeats its own ends.[8] For if space is thus merely a private form of intu

[8] Consult Professor Ladd's *A Theory of Reality;* the chapter on "Space and Motion," especially pages 235-240.

ition, having no correlate in objective reality, it is certainly possible that other beings may have other space-forms without suspecting any difference in the world which they arrange under them. Therefore Euclid's validity, instead of being shown in this way to be a necessity, could only be established by an empirical investigation of the nature of the actual forms of space-intuition of a large number of individuals.[9]

Let us go back now to answer the question which led to this long consideration. The position is now certainly justified that the comparative simplicity and convenience of Euclid is not due to anything essential to intelligence as such, nor is the fact that Euclid is more amenable to algebraic interpretation than his modern rivals to be regarded as a necessity of any rational nature. That certain algebraic formulæ possess greater simplicity than others and that Euclid as a system builds itself up in the mind more easily and readily from certain points of starting than from others are facts which seem native to our minds in a much truer sense than the ordinary facts of sense-perception; but that these facts are *a priori* in the sense of being necessary presuppositions of rationality itself and therefore essential to the very possibility of any world whatever must certainly be denied. For from this purely rational point of view

[9] Compare Lotze's *Metaphysics*, Book II., Chap. 2.

strictly taken there exists no reason why one point of setting out should be regarded as better or more convenient than another. The rational goal in each case is simply to construct the entire system, and when a sufficient number of conditions are postulated to define completely what is logically involved, no room for a preference is left. No difference being admitted in the nature of the entities employed none could be allowed in the relations between them, and it ought to be just as simple and convenient to trace these relations from one point of starting as from another.

It is not meant, of course, that no more than this is essential to a world as actually known, nor is it maintained that a merely formal world could really exist or be truly known if it did exist. That a world of real beings each with peculiar forms and laws of its own must be postulated in the interests of a genuine cognition will be readily granted; *that* they are is a presupposition necessary to the possibility of experience, but precisely *what* they are in particular, is a matter which experience itself must reveal. Hence just what algebraic formulæ, what geometry, and what points of starting in any particular geometry, are really most convenient and simple cannot be foretold by any mere analysis of what it means to be rational. These are truths which are essentially *a posteriore* in their character; they come from experience, individual and racial,

and hence to experience itself the ultimate appeal must be made to tell what they are.

We may certainly conclude then that the parallel postulate and the corresponding assumptions of the other geometries are competitive possibilities arising out of the difference between the exactitude demanded by mathematics and the approximate precision which the limitations of sense-perception inevitably impose. The question as to which possibility, if any one of the group, exactly accords with reality, is one which can be answered only when the nature of reality is itself more accurately known. The validity of Euclid is therefore, at bottom, an empirical matter which can be decided, so far as decision is possible, only by a more refined appeal to actual sense-experience.

The scientific question as to how this appeal can best be made has frequently come to the surface in what has been said, but a fuller reply to it must now be attempted. And first let account be taken of the peculiar difficulties which beset even the best evidence we have concerning the nature of actual space. The Euclidean conception is approximately correct. This we know. The facts thus far do not necessitate the least departure from it, and yet it is possible, though not very probable, that such anomalous facts may at length be found. But if they should be found, there is every consideration to dispose one to interpret them, if possible, in har-

mony with Euclid. In the actual measurements of astronomical distances, and in the formulation of the laws of optics, physics and astronomy the parallel postulate has been confidently accepted. It is furthermore a custom with physicists resulting from long and thoroughly tried practical experience, to adhere steadfastly to the *simplest* assumptions until the facts have so complicated them as to force their rejection or their modification. Now of all the concepts which any physical inquiry must employ, the simplest of course are those of space and time. We can imagine any sort of figure or complication of figures to be realized in Euclidean space without doing violence to this conception. Other concepts are more narrowly restricted by the nature of the facts. A perfect gas, or a perfectly elastic body, for example, does not exist. The physicist is perfectly conscious that these are fictions which conform only approximately and by arbitrary simplifications to the actual facts. There are deviations which cannot be removed. Hence, among the conceptions of physics there is a recognized scale of perfection; and when facts are discovered that render a modification of some sort necessary it is naturally the less perfect conceptions which must suffer the change. Consequently if facts should be discovered which seemingly go against Euclid, it would be simpler and easier to maintain that the laws of physics, astronomy or

optics are slightly incorrect than to sacrifice Euclid and reorganize the whole of physical science upon the basis of a different geometry.

It is therefore obvious that mere measurements of stellar parallax, however refined, can never be regarded as conclusive evidence either for or against Euclid. Under present instrumental limitations and since the base line cannot in any event be greater than the diameter of the earth's orbit, the probable error of the best astronomical observations admits the possibility of a departure of a triangle's angle sum from two right angles, amounting to ten degrees or more in the case of stellar triangles whose sides all equal the distance from the earth to Sirius. Such a measurement, assuming as it does that rays of light from the most distant stars are Euclidean straight lines, is far from proving that K is infinite. It only shows that the actual space constant is very large as compared even with stellar distances.

For short measurements on the earth rays of light are perhaps the best examples we have of actual straight lines, and we naturally assume that they maintain this characteristic even in the most remote regions of space. We must make some assumption regarding the course of light in these far off regions, otherwise astronomical measurements are altogether impossible. In doing so we are obliged to infer that what holds true of light here also holds true of it there, but we have seen

that within the limits of fact which must serve as a basis for this inference Euclidean and non-Euclidean straight lines are not to be distinguished from each other. Hence with K finite but very large as compared with terrestrial measurements and with different laws of optics the same facts might be accounted for in a manner quite as simple as they are at present.

On the other hand, the discovery of a parallax in the case of the most distant star or of a negative parallax for any star cannot be regarded as disproving Euclid. Such anomalies, if not too numerous, could be accounted for in a manner which is much less expensive by making suitable changes in the received laws of optics; or even by assuming a slight strain in the ether itself in those particular regions of space. While therefore an actual discovery of any such facts would certainly prove interesting and suggestive this alone could not be accepted as a conclusive proof or disproof of Euclid's validity; for the very conditions which would render this discovery possible must be found to involve certain assumptions which might very well be withdrawn. The facts, therefore, which count most either for or against Euclid must be found if possible within the realm of direct observation where such assumptions as to what takes place beyond this realm do not need to be made. The problem then is at bottom a psychological one

which must be decided for the most part by experiment. What are the sensory contributions which should be taken into account in the formation of the abstract conception of space? If we confine ourselves to the contributions of vision alone our geometry is not Euclidean; it is projective, and so far as its space conception is concerned is not to be distinguished from the double or single elliptic systems of metrical geometry. But if we consider the larger realm of spatial experiences in which sensations of motion, direction and touch are also involved, all the known facts inevitably suggest the parallel postulate. Whether they shall continue to do so as tested by the most refined experimental analysis of man's ability to discriminate under the most favorable conditions slight variations in the size of angles, in the length of lines, and in the latter's departure from ideal straightness remains to be determined.

As already stated, the author has undertaken an experimental investigation of this sort, at the suggestion of Professor Ladd. Unfortunately the data thus far obtained are not sufficient to justify any positive statement as to what the outcome will probably be. Obviously a very great number and variety of experiments need to be performed by a large number of individuals to obtain results that can be regarded as significant.

CONCLUSIONS

AS TO THE

NATURE OF SPACE.

CHAPTER VI.

RESULTING IMPLICATIONS AS TO THE NATURE OF SPACE.

In the present chapter it shall be our purpose to point out certain implications as to the nature of space which seem to result from denying the necessary validity of the parallel postulate and from the consequent possibility of non-Euclidean geometries. As a preparatory step in this direction it is necessary to get a clear understanding of precisely what is meant by non-Euclidean spaces. The question therefore recurs, are these so-called spaces after all anything more than ingenious logical constructs which one cannot even think of except as finite forms in Euclidean spaces?

In framing a reply to this question we must guard against certain errors which almost inevitably result from the introduction of Euclidean analogies of corresponding non-Euclidean conceptions. Spherical and pseudo-spherical surfaces upon which non-Euclidean straight lines are represented as curves in ordinary Euclidean space of three dimensions are, in reality, very different from

the corresponding non-Euclidean two-dimensional spaces which they are designed to represent. These surfaces are not non-Euclidean spaces, although the analytical treatment is the same in both cases. They are merely analogies to aid the imagination. Failure to understand this distinction has often led to the error of regarding non-Euclidean two-dimensional spaces as curved surfaces in tri-dimensional Euclidean space, and non-Euclidean spaces of three dimensions as constructs requiring a Euclidean space of four dimensions for their possible realization.

If this view were correct it would be very easy to solve the special philosophical problem by reducing everything to Euclid. A non-Euclidean space of any number of dimensions gets itself realized in a Euclidean space of higher dimensions, non-Euclidean planes are Euclidean curved surfaces, and non-Euclidean straight lines are Euclidean curves; hence all we need is a vocabulary whereby the language of one system may be translated into that of another, and the problem is very beautifully solved. Non-Euclidean geometry thus becomes a mere play upon words; it is simply Euclid with a change of names. The next step is to argue that Euclidean spaces of four or more dimensions are inconceivable, and consequently that non-Euclidean spaces of even three dimensions are impossible. This done, the refutation of non-Euclid would stand complete.

But this solution of the difficulty would be altogether too easily won. There seems to be an error involved, and we need to discover its source. The confusion springs, it seems to me, from a misapprehension of the proper meaning of space and of curvature as applied to space. To dispel the fog we must clear up the meaning of these two conceptions.

Space, properly speaking, always means a totality. There is considerable difference between a Euclidean two-dimensional space, for example, and the Euclidean plane as ordinarily conceived. Their internal relations are exactly the same; their analytical treatment is also the same. But the plane has in addition to these internal relations certain external relations; it is two-sided, for instance, and has position. A two-dimensional space, on the other hand, is a *totality,* it contains everything within itself. It therefore has no position and is not two-sided.

In the ordinary study of Euclidean surfaces we frequently make use of certain points and figures which are not on these surfaces at all. We talk of normals, of tangent lines and planes; we say that surfaces may be flexed, shoved about, turned over, or displaced in any direction or into any position. In the proof of equality by the principle of congruent superposition, it is permitted to take a figure from off the plane into a third dimension, obvert it and bring it back again into congruence with another figure in the plane. In a two-dimensional space none of these

things are possible; space, as we have said, must be complete in itself and independent of everything else. But the self-dependence of any space can only be guaranteed when it is shown that measurements can be effected and a complete geometry constructed without leaving it at all or even appealing to anything external to it. In other words, it must be possible for intelligent beings inhabiting the space in question, capable of moving about in it and conscious of nothing at all outside, to construct a geometry true for any and every part of it.

As we have seen, a Euclidean plane may be wrapped about a cylinder or a cone, it may be transformed in a thousand ways so far as its relations to a third dimension are concerned, it may even be wadded into a confused mass and yet so long as no straining or distortion of its internal relations occurs its own proper geometry remains unchanged. Hence considered as a two-dimensional space it is the same throughout. For intelligent beings whom we have supposed to inhabit it, who cannot leave it and who know nothing external to it, it is just the same space; its metrical properties have not changed, all its figures remain the same, and its straight lines are always and everywhere the same straight lines.

That the construction of such a geometry is possible must be admitted when Riemann's formula for *ds* comes to be clearly understood. The simple conditions which we have imposed, that the curvature of

the space in question shall be constant and that its internal relations shall remain intact, admit the existence of equal spatial quantities in different places, and this is all that Riemann's arc formula requires. By selecting co-ordinates which shall have meaning in the space with which we are dealing, and this can always be done, we shall have all the conditions necessary to the analytical construction of the required geometry.

The same truth obviously holds for non-Euclidean spaces of two dimensions, for in these also the necessary conditions of free mobility and of unchanged internal relations are fully supplied. Two-dimensional manifolds, both Euclidean and non-Euclidean, considered individually as self-dependent totalities, are therefore possible, and we may also have many varieties of these provided we think of them as unbounded totalities without external relations. We need, then, simply to keep in mind the fact that a surface in a space of any variety of curvature is quite a different thing from a two-dimensional space of that same variety. The surface always requires a third dimension; the two-dimensional space does not.

In the case of tri-dimensional spaces the same principle holds good. Here, too, more than one variety of space is possible; but in this case the problem for imagination is an impossible one. As tri-dimensional beings, we may easily represent to ourselves

ordinary surfaces as two-dimensional spaces for rational beings supposed to inhabit them and to be conscious only of their internal relations. We can then see from the standpoint of our third dimension the numerous modifications which can be made to take place in the external relations of these surfaces without in any sense disturbing their internal relations. Under the conditions imposed the two-dimensional beings inhabiting these spaces, obviously, could not become aware of any of these external changes; but supposing them to be endowed with intellects like our own, some Lobatchewsky among them would sooner or later work out two-dimensional geometries which would not hold for the particular space inhabited by him. Three-dimensional geometries might also be constructed by this same two-dimensional genius. And yet such a being could not have the necessary sensory experience to represent to himself what these spaces really are as judged by their external relations.

As we stand externally related to these two-dimensional beings, so it is conceivable a four-dimensional being might stand related to us. For such a being various modifications in the external relations of the space-world known to us and of the objects in it might occur without our being at all conscious of the change.

We cannot in any *a priori* fashion dogmatically deny the existence of a four-dimensional space-world

any more than our two-dimensional beings could deny that our world exists. That such a world is inconceivable in the sense that it is impossible for *us* to imagine it must be admitted on the ground of actual experience. We have not had the peculiar sense-experience in a fourth dimension which could render any sort of picture of it possible. We can, of course, say with confidence that our universe as we know it and every known agency in it is confined by some law of its being or at least of our knowing to three dimensions. Our space, so far as actual sense-experience goes, is undoubtedly tri-dimensional. But we know this not as an *a priori* necessity, as Kant contended, but as an empirical fact. We cannot, therefore, rightly affirm that spaces of higher dimensions than ours are objectively impossible or even inconceivable in the sense that they are not rationally construable.

Did a four-dimensional world exist externally related to our own as ours, in turn, stands related to the two-dimensional world just considered, then, as we have said, it would be possible for an inhabitant of that world to witness a vast number of external modifications of our world without our being at all conscious of any change, for no intrinsic transformation would actually occur. The known world would remain to us the same spatially unchanging reality that it now appears to be.

But if we deny the actuality of a fourth dimen-

sion, the modifications which we have supposed to take place in that dimension, of course, could not occur. In that event would more than one variety of tri-dimensional space be possible? This question is a crucial one. Its answer determines whether or not non-Euclidean geometry is to be regarded as having any peculiar philosophical significance. The world, as our actual experience reveals it, is certainly tri-dimensional; judged by the same standard, it is also Euclidean. If, then, only one variety of tri-dimensional space is possible, if non-Euclidean tri-dimensional geometry really demands a fourth dimension, the so-called non-Euclidean spaces are in reality not spaces at all, for they are not self-dependent totalities. It is not, then, a question as to whether non-Euclidean geometrics are possible, but a question as to whether non-Euclidean tri-dimensional spaces are possible. It is, of course, possible to construct such geometries by making use of the idea of a fourth dimension, just as we ordinarily build up our plane geometry by frequently referring to figures which are only possible in a third dimension; but this, of course, is very different from establishing the possibility of non-Euclidean tri-dimensional spaces.

The question, then, simply reduces to this: Are tri-dimensional space-worlds rationally possible whose internal relations considered as totalities are essentially different from each other? And it is an-

swered by showing that the geometries of such spaces can be constructed without appealing to a fourth dimension. This can be done. As in the case of two-dimensional spaces, we have here also all the conditions necessary to render such geometries possible. Indeed, the most interesting and significant feature of non-Euclidean solid geometries lies in the fact that they are just as independent of a fourth dimension as is Euclid itself. There are, to be sure, certain facts in all these geometries that make us wish sometimes for a fourth dimension and the power of moving into it, but they do not necessarily imply this dimension. The simple principle of congruence fails, for example, if we attempt to apply it directly in proving the equality of two Euclidean pyramids whose corresponding parts are mutually equal but arranged in reverse order. As we have said, the analogous theorem in plane geometry is proved by obverting one of the triangles in the third dimension. Were there a fourth dimension and had we the power of moving into it, it is conceivable that this might also be done for the pyramids. What would happen is simply this: By obverting one of the pyramids in the fourth dimension and then returning it to its own tri-dimensional world, its relations to the other objects of this world are changed in a way that is wholly impossible so long as we confine it to three dimensions. But the internal relations of the pyramid itself, as in the observed case of

the triangle, remain entirely unaltered. The self-identity of the figure is retained. But as we have said, these facts cannot be regarded as implying the logical dependence of Euclid, or of non-Euclid, upon a fourth dimension.

Granting, then, as we must, the possibility of internally different tri-dimensional spaces, we need to inquire briefly into their mutual relations and what they imply.

Their differences, as already indicated, are chiefly metrical. Hence in all of them those qualitative characteristics which are presupposed in measurement are implied. It is these characteristics which render possible in each individual case those peculiar internal relations which constitute the essential nature of space as a totality. Whatever intrinsic qualities are necessary to all measurement, as such, these different spaces must possess in common. Just what their common elements are we shall postpone for the moment and consider here their differences merely. These differences are bound up in the modifications of meaning which are possible in the peculiar conception of curvature to which we have often referred.

As already pointed out,[1] curvature as applied to space represents an intrinsic property of the particular space in question and does not necessarily imply

[1] Chap. II., p.

a higher dimension. Measurement in the exact sense required by geometry is not possible unless the space in which it occurs possesses the property of perfect homogeneity. It must permit the free mobility of a rigid standard, and its metrical peculiarities, whatever they are, must be true of it as a whole. This is what is meant by saying that its curvature is constant.

If, then, geometrical spaces of the same number of dimensions may be of different varieties, it follows that their space-constants must also be different, and it is this difference that we need to examine. It is almost universally regarded as a quantitative one. Men talk as though space constants might be arranged as a series of quantities whose ratios could be exactly determined by the use of a common standard of measurement. This is wrong. The law of exact homogeneity forbids it. Strictly speaking, quantitative differences can only exist in the *same* space, for here alone the exact measurement of all figures by the same rigid standard is possible. Different spaces of the same number of dimensions do not admit of the free mobility throughout their whole extent of the same rigid body; if they did they could not be metrically distinguished. But, as we have said, it is in metrical properties that Euclidean and non-Euclidean space-conceptions are different; projectively treated these differences disappear. Space-constants must therefore be regarded as stand-

ing related to each other in a manner similar to what obtains in different shades of the same color. They are qualitatively different. They have much in common, but are at the same time so unlike as to make exact geometrical measurement impossible. There exists no common standard; hence, strictly speaking, we cannot say that any space-constant is greater or less than another. Logical difficulties of a similar nature obviously follow the attempt to regard the curvature of any geometrical space as a variable.[2] This involves the assumption of an absolute position and consequently denies the relativity and homogeneity which strict geometrical measurement always requires. Such a variable curvature is represented for example by the surface of an egg, in which, generally speaking, it is not possible to make one piece fit upon another without distortion. Regarded as a space, it is not homogeneous.

Here again we are forced to observe the necessary disparity between the facts of experience and the ab-

[2] It is not necessary to enter into an extended argument in support of this position. The curvature of space might be regarded as variable if the law of its change were known. This is practically what is involved in Cayley's idea of constructing non-Euclidean geometries in Euclidean spaces by simply modifying the notion of distance. It is also suggested by Erdmann (*Die Axiome der Geometrie,* Leipzig, 1877). But while such a law of change may be assumed we are left absolutely without any means of determining what it is or even of detecting its presence. Differences of magnitude where comparison can not reveal them are manifestly at variance with the very notion of magnitude.

solute exactitude demanded by geometry. Geometrical measurement is of course impossible without the assumption of absolute homogeneity; but practical measurement, the only kind possible in the world as we know it, can be performed. It is also clear, in support of our general position, that the question whether the actual space-parameter of our world, when assumed to be constant, is Euclidean or non-Euclidean, cannot be settled by any mere appeal to a rigorous geometry. For, as we have just seen, the metrical comparison of different space-parameters upon such a basis cannot occur. We cannot travel, so to speak, with our metrical standard from one space to another.

We must remember, however, that spatial homogeneity is merely a conception; as to its nature and origin it does not differ from other conceptions. It is simply an idealization of certain familiar facts of our spatial experience. It is the facts that determine the meaning of homogeneity, and not homogeneity that determines the facts. Geometry must fit experience, not experience geometry.

To reverse this order is a crime against knowledge which has been too often committed. The homogeneity of space is in fact a complex idea and admits of being variously interpreted. This is shown by the existence of different metrical geometries in which homogeneity is so conceived as not to admit of the free mobility of the same rigid body in the

different cases. It is not, then, a question as to which of these conceptions is to be accepted as the absolute truth, but as to which one accords, and will continue to accord, best with man's actual experience. Experience itself must decide this matter, and even its decision must always be regarded as only approximate. Homogeneity, then, whatever particular view we take of it, is a different conception from mere logical *anyness* with which it is often confused. It bears, in some sense, the stamp of actual experience as is seen in the fact that the geometrical figures which it admits appear to have a distinct character and reality of their own. If one does not believe this let him take any material body and endeavor to cut from it a Euclidean regular polyhedron of seven plane faces and his doubts will vanish. But why is this figure impossible? It is certainly not because of any obstacles which the material itself can offer, for this may be sawed through in one direction as easily as another. The truth is the lines, angles, and planes enter into the structure of such geometrical figures with all the meaning which our actual tri-dimensional experience has given them. Indeed, it ought now to be perfectly clear that lines and planes as ordinarily conceived are not, in reality, one and two-dimensional objects respectively. They are this and more. To give them their full meaning they require a spatial experience of a tri-dimensional order. The straight line

as it exists in the average man's consciousness is perhaps best defined as a line which looks the same from every point not in it, but this definition clearly implicates three dimensions. If, however, we define it as the shortest distance between two points, and it is then pointed out that arcs of great circles on a sphere meet this requirement, we are not satisfied; for this is evidently not what we mean. It is the chord, we say, not the arc, that stands as a true representative of our idea of the straight line. Thus again we must enter the third dimension before we can decide the matter. If an ordinary plane be conceived as wrapped about a cone, it remains the same two-dimensional object that it was before, but it is for us no longer a plane. To be a plane it must sustain certain definite relations to a third dimension.

It is therefore obvious that the peculiar form of our tri-dimensional experience has given us these conceptions of the straight line and the plane. The moment we begin to deal with definite geometrical figures of any sort we are considering combinations of relations whose peculiar character as combinations is determined by certain conceptions which are thus grounded in experience. They are certainly not rational necessities. Homogeneity of some sort is an essential characteristic of any space-conception which will admit of the exact quantitative determinations of metrical geometry. The particular combinations of this conception and those assumptions which are

needed in each case to define accurately and completely the peculiar foundations of the different geometries must, of course, be regarded as necessary to the particular space-conceptions upon which these geometries are founded; but necessities of thought, presupposed in the very possibility of any spatial experience whatever or even of experience as we actually have it, they certainly are not.

Underneath these conceptions and implicit in them the true category of space is to be found. As a "differentiating principle"[3] of some sort, both subjectively and transsubjectively operative, externalizing and at the same time uniting a world of co-existing selves and things, this category is necessary to all human cognition or mental representation of the world of reality. As such we cannot escape it, for we can think of no entity to which it does not apply. It was the unavoidable presence of space in this meaning of the word that seems to contradict flatly much that was said at the beginning of this chapter, where we represented geometrical spaces as totalities and yet spoke of them as though they stood related somehow to realities to which as spaces they could not apply. The confusion at that stage of the discussion seemed necessary. We could not avoid it. It is now possible to make clear what was meant. The space-conceptions of geometry are not catego-

[3] See Professor Ladd's admirable chapter on "Space and Motion," in *A Theory of Reality*, New York, 1899, pp. 214-252.

ries, not necessities of thought or reality; they imply the space category, it is true, but they also imply more. Stamped upon them are certain marks of an empirical origin which are grounded in the peculiar nature of our human spatial experience. To determine these marks which, so far as we can see, might have been different; to separate them out and if possible to get directly at the peculiar nature of the category itself, was the difficult problem which we there had in mind. When we limit the number of dimensions and introduce particular figures and special metrical considerations we no longer deal with the "pure" category of space, as it must lie implicit in any rational being.

We conclude, then, that the actual characters of our space experience so far as they involve tri-dimensionality, the parallel postulate, &c., are empirical in the same sense in which any notable feature of our present human experience is empirical. As it is an empirical fact that there are many different men with different minds or that we can grasp in one act of attention only a narrow range of facts, so it is an empirical fact that the above-mentioned characters belong to our space-consciousness. Other beings may have other space experiences. We ourselves may some day acquire other such experiences. There is no demonstrable necessity accessible regarding the matter, so far as the details are concerned.

The only *a priori* manifold at present definable in *Kant's sense of a priori* seems to be a manifold constituted by a totality of logical classes or distinctions of any similar sort. The constitution of such a complete system of logical entities must be implicitly known to any rational being. It depends upon the fundamental illative relation expressed by such statements as "a implies b," or that "a is subsumed under b." A world in which this relation has full sway contains the "negation" of any of its own terms. It contains series of subsumptions and also certain combinations whereby its terms are grouped into wholes. Such a system has a formal structure which Kempe[4] has shown includes all the types of order which appear in every sort of geometry of *n* dimensions. Any space you please may be viewed as a selection from amongst the entities of such a system made by adding certain arbitrary postulates to these fundamental logical ones.

The connection between this *a priori* logical manifold and the empirical space of our own experience lies in the fact that the space-aspect of experience is the one which most definitely implies and is implied by our power to co-ordinate our activities so that "a leads to b leads to c," &c. It is that aspect which enables us to introduce illative relations

[4] "Relation Between the Logical Theory of Classes and the Geometrical Theory of Points." Proceedings of the London Mathematical Society for 1890, Vol. 21.

among acts and systems of acts of our own (acts actual and acts possible).

That this aspect of experience exists is an empirical fact. *What* correlations of acts it permits and *how* it permits them are also empirical. All the details are empirical. But if it is to permit such a system at all, it has to conform to the general type of the illative relation and its parts viewed as co-existent must be related to each other in accordance with the general type of an illative relation. "No actualization of the space-principle is, therefore, possible either from the point of view of its subjective origin or of its trans-subjective applicability, unless this principle itself is conceived of as *the mode of the action of one all-differentiating and yet all unifying Force,"* [5] functioning in a way that is essentially law-abiding and rational.

[5] Ladd's *A Theory of Reality,* p. 252.

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